

Decoupling of EYMH Equations, Off-Diagonal Solutions, and Black Ellipsoids and Solitons

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Abstract

This paper is concerned with giving the proof that there is a general decoupling property of vacuum and nonvacuum Einstein equations written in variables adapted to nonholonomic 2+2 splitting. We show how such a geometric techniques can be applied for constructing generic off-diagonal exact solutions of Einstein-Yang-Mills-Higgs (EYMH) equations. The corresponding classes of solutions are determined by generating and integration functions which depend, in general, on all space and time coordinates and may possess, or not, Killing symmetries. The initial data sets for the Cauchy problem and their global properties are analyzed. There are formulated the criteria of evolution with spacetime splitting and decoupling of fundamental field equations. Examples of exact solutions defining black ellipsoid and solitonic configurations are provided.

Keywords: Einstein spaces, gauge and scalar fields in general relativity, exact solutions, Cauchy problem.

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1 Introduction

The issue of constructing exact and approximate solutions of gravitational and matter field equations is of interest in mathematical relativity, particle physics and for various applications in modern cosmology and astrophysics. The Einstein, Yang–Mills, Higgs and other fundamental field equations in physics are sophisticated systems of nonlinear partial differential equations (PDE) which are very difficult to be integrated and studied in general forms. There were elaborated various geometric, analytic and numeric methods, see [1, 2, 3, 4, 5, 6, 7, 8] and references therein.

In this work, we address again and develop a geometric approach (the so-called anholonomic deformation method, see [9, 10, 11]) to constructing exact solutions of PDE describing generic off-diagonal and nonlinear gravitational, gauge and scalar field interactions. The goal is to provide new results on decoupling property¹ of Einstein–Yang–Mills–Higgs, EYMH, equations and integration of such systems. Certain issues on the Cauchy problem and decoupling of Einstein equations and the initial data sets and nonholonomic evolution (with non-integrable constraints and/or with respect to anholonomic frames) will be analyzed. Examples of off-diagonal solutions for EYMH black holes/ellipsoids and solitonic configurations will be given.

In Section 2 we provide the main Theorems on decoupling the Einstein equations for generic off-diagonal metrics with one Killing symmetry and consider extensions to classes of "non-Killing" solutions with coefficients depending on all set of four coordinates. We prove also that the decoupling property also holds true for certain classes of nonholonomic EYMH systems. The theorems on generating off-diagonal solutions are considered in Section 3. Section 4 contains a study of the Cauchy problem in connection to the decoupling property of Einstein equations. Next two sections are devoted to examples of generic off-diagonal exact solutions of the EYMH equations. In Section 5 there are constructed nonholonomic YM vacuum deformations of black holes. Ellipsoid-solitonic non-Abelian configurations are analyzed in Section 6. Appendix A contains a survey on the geometry of nonholonomic 2+2 splitting of Lorentz manifolds. The most important formulas and computations which are necessary to prove the decoupling property of Einstein equations are given in Appendix B.

¹it is used also the term "separation" of equations, which should not be confused with separation of variables

2 Decoupling Property of Einstein Equations

In general relativity (GR), the curved spacetime (V, \mathbf{g}) is defined by a pseudo-Riemannian manifold V endowed with a Lorentzian metric \mathbf{g} as a solution of Einstein equations.² The space of classical physical events is modelled as a Lorentzian four dimensional, 4-d, manifold \mathcal{V} (of necessary smooth class, Hausdorff and paracompact one) when the symmetric 2-covariant tensor $\mathbf{g} = \{g_{\alpha\beta}\}$ defines in each point $u \in \mathcal{V}$ and nondegenerate bilinear form on the tangent space $T_u\mathcal{V}$, for instance, of signature $(+++-)$. The assumption that $T_u\mathcal{V}$ has its prototype (local fiber) the Minkowski space $\mathbb{R}^{3,1}$ leads at a causal character of positive/negative/null vectors on \mathcal{V} , i.e. for the module \mathcal{X} of vectors fields $X, Y, \dots \in \mathcal{X}(\mathcal{V})$, which is similar to that in special relativity.

Let us denote by e_α and e^β a local frame and, respectively, its dual frame [we can consider orthonormal (co) bases], where Greek indices α, β, \dots may be abstract ones, or running values 1, 2, 3, 4. For a coordinate base $u = \{u^\alpha\}$ on a chart $U \subset \mathcal{V}$, we can write $e_\alpha = \partial_\alpha = \partial/\partial u^\alpha$ and $e^\beta = du^\beta$ and, for instance, define the coefficients of a vector X and a metric \mathbf{g} , respectively, in the forms $X = X^\alpha e_\alpha$ and

$$\mathbf{g} = g_{\alpha\beta}(u) e^\alpha \otimes e^\beta, \quad (1)$$

where $g_{\alpha\beta} := \mathbf{g}(e_\alpha, e_\beta)$.³ We consider bases with non-integrable (equivalently, nonholonomic/anhomonic) 2 + 2 splitting for conventional, horizontal, h, and vertical, v, decomposition, when for the tangent bundle $T\mathcal{V} := \bigcup_u T_u\mathcal{V}$ a Whitney sum

$$\mathbf{N}: T\mathcal{V} = h\mathcal{V} \oplus v\mathcal{V} \quad (2)$$

is globally defined. Such a nonholonomic distribution is determined locally by its coefficients $N_i^a(u)$, when $\mathbf{N} = N_i^a(x, y) dx^i \otimes \partial/\partial y^a$, where $u^\alpha = (x^i, y^a)$ splits into h-coordinates, $x = (x^i)$, and v-coordinates, $y = (y^a)$, with indices running respectively values $i, j, k, \dots = 1, 2$ and $a, b, c, \dots = 3, 4$.⁴

²We assume that readers are familiar with basic concepts and results on mathematical relativity and methods of constructing exact solutions outlined, for instance, in above mentioned monographs and reviews.

³The summation rule on repeating low-up indices will be applied if the contrary will be not stated.

⁴We note that the 2+2 splitting can be considered as an alternative to the well known 3+1 splitting. The first one is convenient, for instance, for constructing generic off-diagonal solutions and elaborating models of deformation and/or A-brane quantization of gravity, but the second one is more important for canonical/loop quantization etc.

Any spacetime (V, \mathbf{g}) can be equipped with a non-integrable fibred structure (2) and such a manifold is called nonholonomic (equivalently, N-anholonomic). We use "boldface" letters in order to emphasize that certain spaces and geometric objects/constructions are "N-adapted", i. e. adapted to a h-v-splitting. The geometric objects are called distinguished (in brief, d-objects, d-vectors, d-tensors etc). For instance, we write a d-vector as $\mathbf{X} = (hX, vX)$ for a nonholonomic Lorentz manifold/spacetime (\mathbf{V}, \mathbf{g}) .

On a spacetime (\mathbf{V}, \mathbf{g}) , we can perform/adapt the geometric constructions using "N-elongated" local bases (partial derivatives), $\mathbf{e}_\nu = (\mathbf{e}_i, e_a)$, and cobases (differentials), $\mathbf{e}^\mu = (e^i, \mathbf{e}^a)$, when

$$\mathbf{e}_i = \partial/\partial x^i - N_i^a(u)\partial/\partial y^a, \quad e_a = \partial_a = \partial/\partial y^a, \quad (3)$$

$$\text{and } e^i = dx^i, \quad \mathbf{e}^a = dy^a + N_i^a(u)dx^i. \quad (4)$$

Such (co) frame structures depend linearly on N-connection coefficients being, in general, nonholonomic. For instance, the basic vectors (3) satisfy certain nontrivial nonholonomy relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma, \quad (5)$$

with (antisymmetric) nontrivial anholonomy coefficients

$$W_{ia}^b = \partial_a N_i^b, \quad W_{ji}^a = \Omega_{ij}^a = \mathbf{e}_j(N_i^a) - \mathbf{e}_i(N_j^a). \quad (6)$$

Any spacetime metric $\mathbf{g} = \{g_{\alpha\beta}\}$ (1), via frame/coordinate transforms can be represented equivalently in N-adapted form as a d-metric

$$\mathbf{g} = g_{ij}(x, y) e^i \otimes e^j + g_{ab}(x, y) \mathbf{e}^a \otimes \mathbf{e}^b, \quad (7)$$

or, with respect to a coordinate local cobasis $du^\alpha = (dx^i, dy^a)$, as an off-diagonal metric

$$\mathbf{g} = \underline{g}_{\alpha\beta} du^\alpha \otimes du^\beta,$$

where

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b g_{ab} & N_j^e g_{ae} \\ N_i^e g_{be} & g_{ab} \end{bmatrix}. \quad (8)$$

A metric \mathbf{g} is generic off-diagonal if (8) can not be diagonalized via coordinate transforms. Ansatz of this type are used in Kaluza-Klein gravity when $N_i^a(x, y) = \Gamma_{bi}^a(x)y^a$ and y^a are "compactified" extra-dimensions coordinates, or in Finsler gravity theories, see details in [12, 11]. In this work, we restrict our considerations only to the 4-d Einstein gravity theory. The principle of general covariance in GR, allows us to consider any frame/coordinate

transforms and write a spacetime metric \mathbf{g} equivalently in any above form (1), (8) and/or (7). The last mentioned parametrization will allow us to prove a very important property of decoupling of Einstein equations with respect to N-adapted bases (3) and (4).

Via frame/ coordinate transforms $e_\alpha = e_{\alpha'}^{\alpha'}(x, y)e_{\alpha'}$, $\underline{g}_{\alpha\beta} = e_{\alpha'}^{\alpha'}e_{\beta'}^{\beta'}\underline{g}_{\alpha'\beta'}$, a metric \mathbf{g} (7) can be written in a form with separation of v-coordinates and nontrivial vertical conformal transforms,

$$\begin{aligned}\mathbf{g} &= g_i dx^i \otimes dx^i + \omega^2 h_a \underline{h}_a \mathbf{e}^a \otimes \mathbf{e}^a, \\ \mathbf{e}^3 &= dy^3 + (w_i + \underline{w}_i) dx^i, \quad \mathbf{e}^4 = dy^4 + (n_i + \underline{n}_i) dx^i,\end{aligned}\tag{9}$$

were

$$\begin{aligned}g_i &= g_i(x^k), \quad g_a = \omega^2(x^i, y^c) h_a(x^k, y^3) \underline{h}_a(x^k, y^4), \\ N_i^3 &= w_i(x^k, y^3) + \underline{w}_i(x^k, y^4), \quad N_i^4 = n_i(x^k, y^3) + \underline{n}_i(x^k, y^4),\end{aligned}\tag{10}$$

are functions of necessary smooth class which will be defined in a form to generate solutions of Einstein equations.⁵

The aim of this section is to prove that the gravitational field equations in GR, in the vacuum cases and certain very general classes of matter field sources decouple for parametrizations of metrics in the form (10). For convenience, we present in Appendix A a brief review on the geometry of nonholonomic 2+2 splitting of Lorentz manifolds and Einstein equations.

2.1 Splitting of (non) vacuum gravitational field equations

We shall use brief denotations for partial derivatives, $a^\bullet = \partial a / \partial x^1$, $a' = \partial a / \partial x^2$, $a^* = \partial a / \partial y^3$, $a^\circ = \partial a / \partial y^4$. The equations will be written with respect to N-adapted frames of type (3) and (4).

2.1.1 Off-diagonal spacetimes with Killing symmetry

We use an ansatz (9) when $\omega = 1$, $\underline{h}_3 = 1$, $\underline{w}_i = 0$ and $\underline{n}_i = 0$ in data (10) and $\underline{\mathbf{I}} = 0$ for (A.15). Such a generic off-diagonal metric does not depend on variable y^4 , i.e. $\partial / \partial y^4$ is a Killing vector, if $\underline{h}_4 = 1$. Nevertheless, the decoupling property can be proven for the same assumptions but arbitrary $\underline{h}_4(x^k, y^4)$ with nontrivial dependence on y^4 . We call this class of metrics to be with weak Killing symmetry because they result in systems of PDE (A.12) as for the Killing case but there are differences in (A.13) if $\underline{h}_4 \neq 1$.

⁵There is not summation on repeating "low" indices a in formulas (10) but such a summation is considered for crossing "up-low" indices i and a in (9)). We shall underline a function if it positively depends on y^4 but not on y^3 and write, for instance, $\underline{n}_i(x^k, y^4)$.

Theorem 2.1 *The Einstein eqs (A.12) and (A.13) for a metric \mathbf{g} (10) with $\omega = \underline{h}_3 = 1$ and $\underline{w}_i = \underline{n}_i = 0$ and $\underline{\Upsilon} = 0$ in matter source Υ^β_δ (A.15) are equivalent, respectively, to*

$$\hat{R}_1^1 = \hat{R}_2^2 = \frac{-1}{2g_1g_2}[g_2^{\bullet\bullet} - \frac{g_1^\bullet g_2^\bullet}{2g_1} - \frac{(g_2^\bullet)^2}{2g_2} + g_1'' - \frac{g_1'g_2'}{2g_2} - \frac{(g_1')^2}{2g_1}] = -{}^v\Upsilon, \quad (11)$$

$$\hat{R}_3^3 = \hat{R}_4^4 = -\frac{1}{2h_3h_4}[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^*h_4^*}{2h_3}] = -\Upsilon, \quad (12)$$

$$\hat{R}_{3k} = \frac{w_k}{2h_4}[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^*h_4^*}{2h_3}] + \frac{h_4^*}{4h_4}\left(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4}\right) - \frac{\partial_k h_4^*}{2h_4} = 0, \quad (13)$$

$$\hat{R}_{4k} = \frac{h_4}{2h_3}n_k^{**} + \left(\frac{h_4}{h_3}h_3^* - \frac{3}{2}h_4^*\right)\frac{n_k^*}{2h_3} = 0, \quad (14)$$

and

$$\begin{aligned} w_i^* &= (\partial_i - w_i) \ln |h_4|, \partial_k w_i = \partial_i w_k, \\ n_k \underline{h}_4^\circ &= \partial_k \underline{h}_4, n_i^* = 0, \partial_i n_k = \partial_k n_i. \end{aligned} \quad (15)$$

Proof. See Appendix B. \square

Let us discuss the decoupling (splitting) property of the Einstein equations with respect to certain classes of N-adapted frames which is contained in the system of PDE (11)–(15). For instance, the first equation is for a 2-d metric which always can be diagonalized, $[g_1, g_2]$, and/or made to be conformally flat. Prescribing a function g_1 and source ${}^v\Upsilon$, we can find g_2 , or inversely. The equation (12) contains only the first and second derivatives on $\partial/\partial y^3$ and relates two functions h_3 and h_4 . Prescribing one of such functions and source Υ , we can define the second one taking, respectively, one or two derivations on y^3 . The equation (13) is a linear algebraic system for w_k if the coefficients h_a have been already defined as a solution of (12). Nevertheless, we have to solve a system of first order PDE on x^k and y^3 in order to find w_k resulting in zero torsion conditions (15). Such conditions do not allow a complete decoupling because the first equations relate w_i to $H = \ln |h_4|$ via corresponding first order PDE. Nevertheless, it is possible, for instance, to integrate such solutions for any prescribed H (see more details below, in Remark–Example 3.1). The forth equations (14) became trivial for any $n_i^* = 0$ if we want to satisfy completely such zero torsion conditions.⁶ A nontrivial function \underline{h}_4 is explicitly present in the conditions (15). If such restrictions are satisfied, this allows us to eliminate \underline{h}_4 from (13) (see details in the proof of above theorem).

⁶Nontrivial solutions and nonzero torsion configurations present interest in modified theories of gravity, see such examples in Ref. [17]

We conclude that the Einstein equations for metrics with one Killing symmetry can be such way parametrized with respect to N-adapted frames that they decouple and separate into "quite simple" PDE for h-components, g_i , and then for v-components, h_a . The N-connection coefficients also separate and can be defined from corresponding algebraic and/or first order PDE. The "zero torsion" conditions impose certain additional constraints (as some simple first order PDE with possible separation of variables) on N-coefficients and coefficients of v-metric.

In a similar form, we can decouple the Einstein equations for spacetimes with one Killing symmetry on $\partial/\partial y_4$.

Corollary 2.1 *The Einstein eqs (A.12) and (A.13) for a metric \mathbf{g} (10) with $\omega = h_4 = 1$ and $w_i = n_i = 0$ and $\Upsilon = 0$ in matter source Υ^β_δ (A.15) are equivalent, respectively, to*

$$\hat{R}_1^1 = \hat{R}_2^2 = \frac{-1}{2g_1g_2}[g_2^{\bullet\bullet} - \frac{g_1^\bullet g_2^\bullet}{2g_1} - \frac{(g_2^\bullet)^2}{2g_2} + g_1'' - \frac{g_1'g_2'}{2g_2} - \frac{(g_1')^2}{2g_1}] = -{}^v\Upsilon, \quad (16)$$

$$\hat{R}_3^3 = \hat{R}_4^4 = -\frac{1}{2\underline{h}_3\underline{h}_4}[\underline{h}_3^{\circ\circ} - \frac{(\underline{h}_3^\circ)^2}{2\underline{h}_3} - \frac{\underline{h}_3^\circ\underline{h}_4^\circ}{2\underline{h}_4}] = -\underline{\Upsilon}, \quad (17)$$

$$\hat{R}_{3k} = +\frac{\underline{h}_3}{2\underline{h}_4}\underline{w}_k^{\circ\circ} + \left(\frac{\underline{h}_3}{\underline{h}_4}\underline{h}_4^\circ - \frac{3}{2}\underline{h}_3^\circ\right)\frac{\underline{h}_k^\circ}{2\underline{h}_4} = 0, \quad (18)$$

$$\hat{R}_{4k} = \frac{\underline{n}_k}{2\underline{h}_3}[\underline{h}_3^{\circ\circ} - \frac{(\underline{h}_3^\circ)^2}{2\underline{h}_3} - \frac{\underline{h}_3^\circ\underline{h}_4^\circ}{2\underline{h}_4}] + \frac{\underline{h}_3^\circ}{4\underline{h}_3}\left(\frac{\partial_k\underline{h}_3}{\underline{h}_3} + \frac{\partial_k\underline{h}_4}{\underline{h}_4}\right) - \frac{\partial_k\underline{h}_3^\circ}{2\underline{h}_3} = 0, \quad (19)$$

$$\begin{aligned} \text{and } \underline{n}_i^\circ &= (\partial_i - \underline{n}_i) \ln |\underline{h}_3|, (\partial_k - \underline{n}_k) \underline{n}_i = (\partial_i - \underline{n}_i) \underline{n}_k, \\ \underline{w}_k \underline{h}_3^* &= \partial_k \underline{h}_3, \underline{w}_i^\circ = 0, \partial_i \underline{w}_k = \partial_k \underline{w}_i. \end{aligned} \quad (20)$$

Proof. It is similar to that for Theorem 2.1 provided in Appendix B. We do not repeat such computations. \square

Using above Theorem and Corollary and mutual transforms of the systems of equations, we can formulate:

Conclusion 2.1 *The nonlinear systems of PDE corresponding to Einstein equations (A.12) and (A.13) for metrics \mathbf{g} (10) with Killing symmetry on $\partial/\partial y_4$, when $\omega = \underline{h}_3 = 1$ and $\underline{w}_i = \underline{n}_i = 0$ and $\underline{\Upsilon} = 0$ in matter source Υ^β_δ (A.15), can be transformed into respective systems of PDE for data with Killing symmetry on $\partial/\partial y_3$, when $\omega = h_4 = 1$ and $w_i = n_i = 0$ and $\Upsilon = 0$, if $h_3(x^i, y^3) \rightarrow \underline{h}_4(x^i, y^4)$, $h_4(x^i, y^3) \rightarrow \underline{h}_3(x^i, y^4)$, $w_k(x^i, y^3) \rightarrow \underline{n}_k(x^i, y^4)$ and $n_k(x^i, y^3) \rightarrow \underline{w}_k(x^i, y^4)$.*

The above presented method of nonholonomic deformations can be used for decoupling the Einstein equations even some metrics do not possess, in general, any Killing symmetries. The generic nonlinear character of such systems of PDE does not allow us to use a principle of superposition of solutions. Nevertheless, certain classes of conformal transforms for the v-components of d-metrics and nonholonomic constraints of integral varieties give us the possibility to extend the anholonomic deformation method to "non-Killing" vacuum and nonvacuum gravitational interactions. In next two subsections, we analyze two possibilities to decouple the Einstein equations for metrics with coefficients depending on all spacetime coordinates.

2.1.2 Preserving decoupling under v-conformal transforms

This property is stated by

Lemma 2.1 *The Einstein equations (A.12) for geometric data (B.1), i.e. the system of PDE (12)–(14), do not change under a "vertical" conformal transform with nontrivial $\omega(x^k, y^a)$ to a d-metric (10) if there are satisfied the conditions*

$$\partial_k \omega - w_i \omega^* - n_i \omega^\circ = 0 \text{ and } \hat{T}_{kb}^a = 0. \quad (21)$$

Proof. It follows from straightforward computations when coefficients $g_i(x^k), g_3 = h_3(x^k, y^3), g_4 = h_4(x^k, y^3) \underline{h}_4(x^k, y^4), N_i^3 = w_i(x^k, y^3), N_i^4 = n_i(x^k, y^3)$ are generalized to a nontrivial $\omega(x^k, y^a)$ with ${}^\omega g_3 = \omega^2 h_3$ and ${}^\omega g_4 = \omega^2 h_4 \underline{h}_4$. Using respectively formulas (A.2), (A.3), (A.6) and (A.7), we get distortion relations for the Ricci d-tensors (A.11),

$${}^\omega \hat{R}_b^a = \hat{R}_b^a + {}^\omega \hat{Z}_b^a \text{ and } {}^\omega \hat{R}_{bi} = \hat{R}_{bi} = 0,$$

where \hat{R}_b^a and \hat{R}_{bi} are those computed for $\omega = 1$, i.e. (12)–(14). The values ${}^\omega \hat{R}_b^a$ and ${}^\omega \hat{Z}_b^a$ are defined by a nontrivial ω and computed using the same formulas. We do not repeat here such details provided in Refs. [10, 11] for $\underline{h}_4 = 1$ because a nontrivial \underline{h}_4 does not modify substantially the proof that ${}^\omega \hat{Z}_b^a = 0$ if the conditions (21) are satisfied. \square

Using Theorem 2.1, Corollary 2.1, Conclusion 2.1, Lemma 2.1, we prove

Theorem 2.2 *A d-metric*

$$\begin{aligned} \mathbf{g} &= g_i(x^k) dx^i \otimes dx^i + \omega^2(x^k, y^a) (h_3 \mathbf{e}^3 \otimes \mathbf{e}^3 + h_4 \underline{h}_4 \mathbf{e}^4 \otimes \mathbf{e}^4), \\ \mathbf{e}^3 &= dy^3 + w_i(x^k, y^3) dx^i, \quad \mathbf{e}^4 = dy^4 + n_i(x^k) dx^i, \end{aligned} \quad (22)$$

satisfying the PDE (11)–(15) and $\partial_k \omega - w_i \omega^* - n_i \omega^\circ = 0$, or a d -metric

$$\begin{aligned} \mathbf{g} &= g_i(x^k) dx^i \otimes dx^i + \omega^2(x^k, y^a) (h_3 \underline{h}_3 \mathbf{e}^3 \otimes \mathbf{e}^3 + \underline{h}_4 \mathbf{e}^4 \otimes \mathbf{e}^4), \\ \mathbf{e}^3 &= dy^3 + \underline{w}_i(x^k) dx^i, \quad \mathbf{e}^4 = dy^4 + \underline{n}_i(x^k, y^4) dx^i, \end{aligned} \quad (23)$$

satisfying the PDE (16)–(20) and $\partial_k \omega - \underline{w}_i \omega^* - \underline{n}_i \omega^\circ = 0$, define, in general, two different classes of generic off-diagonal solutions of Einstein equations (A.12) and (A.13) with respective sources of type (A.15).

Both ansatz of type (22) and (23) consist particular cases of parametrizations of metrics in the form (9). Via frame/coordinate transform into a finite region of a point ${}^0u \in \mathbf{V}$ any spacetime metric in GR can be represented in an above mentioned d -metric form. If only one of coordinates y^a is time-like, the solutions of type (22) and (23) can not be transformed mutually via nonholonomic frame deformations preserving causality.

2.1.3 Decoupling with effective linearization of Ricci tensors

The explicit form of field equations for vacuum and nonvacuum gravitational interactions depends on the type of frames and coordinate systems we consider for decoupling such PDE. We can split such systems for more general parameterizations (than ansatz (22) and (23)) in a form (9) with nontrivial ω and N -coefficients in (10). This is possible in any open region $U \subset \mathbf{V}$ where for computing the N -adapted coefficients of the Riemann and Ricci d -tensors, see formulas (A.6) and (A.7), we can neglect contributions from quadratic terms of type $\hat{\Gamma} \cdot \hat{\Gamma}$ but preserve values of type $\partial_\mu \hat{\Gamma}$. For such constructions, we have to introduce a class of N -adapted normal coordinates when $\hat{\Gamma}(u_0) = 0$ for points u_0 , for instance, belonging to a line on U . Such conditions can be satisfied for decompositions of metrics and connections on a small parameter like it is explained in details in Ref. [9] (see decompositions on a small eccentricity parameter ε in Section 5). Other possibilities can be found if we impose nonholonomic constraints, for instance, of type $h_4^* = 0$ but for nonzero $h_4(x^k, y^3)$ and/or $h_4^{**}(x^k, y^3)$; such constraints can be solved in non-explicit form and define a corresponding subclass of N -adapted frames. Considering further nonholonomic deformations with a general decoupling with respect to a "convenient" system of reference/coordinates, we can deform the equations and solutions to configurations when terms of type $\hat{\Gamma} \cdot \hat{\Gamma}$ became important.

Theorem 2.3 ("Non-quadratic" decoupling) *The Einstein equations in GR (for instance, in the form (A.12) and (A.13)), via nonholonomic*

frame deformations to a metric \mathbf{g} (10) and matter source Υ^β_δ (A.15), when contributions from terms of type $\hat{\Gamma} \cdot \hat{\Gamma}$ are considered small for an open region $U \subset \mathbf{V}$, can be transformed equivalently into a system of PDE with h - v -decoupling:

$$\hat{R}_1^1 = \hat{R}_2^2 = \frac{-1}{2g_1g_2}[g_2^{\bullet\bullet} - \frac{g_1^\bullet g_2^\bullet}{2g_1} - \frac{(g_2^\bullet)^2}{2g_2} + g_1'' - \frac{g_1'g_2'}{2g_2} - \frac{(g_1')^2}{2g_1}] = -{}^v\Upsilon, \quad (24)$$

$$\begin{aligned} \hat{R}_3^3 &= \hat{R}_4^4 = -\frac{1}{2h_3h_4}[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^*h_4^*}{2h_3}] - \frac{1}{2\underline{h}_3\underline{h}_4}[\underline{h}_3^{\circ\circ} - \frac{(\underline{h}_3^\circ)^2}{2\underline{h}_3} - \frac{\underline{h}_3^\circ\underline{h}_4^\circ}{2\underline{h}_4}] \\ &= -\Upsilon - \underline{\Upsilon}, \end{aligned} \quad (25)$$

$$\begin{aligned} \hat{R}_{3k} &= \frac{w_k}{2h_4}[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^*h_4^*}{2h_3}] + \frac{h_4^*}{4h_4}\left(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4}\right) - \frac{\partial_k h_4^*}{2h_4} \\ &\quad + \frac{\underline{h}_3}{2\underline{h}_4}\underline{n}_k^{\circ\circ} + \left(\frac{\underline{h}_3}{\underline{h}_4}\underline{h}_4^\circ - \frac{3}{2}\underline{h}_3^\circ\right)\frac{\underline{n}_k^\circ}{2\underline{h}_4} = 0, \end{aligned} \quad (26)$$

$$\begin{aligned} \hat{R}_{4k} &= \frac{w_k}{2\underline{h}_3}[\underline{h}_3^{\circ\circ} - \frac{(\underline{h}_3^\circ)^2}{2\underline{h}_3} - \frac{\underline{h}_3^\circ\underline{h}_4^\circ}{2\underline{h}_4}] + \frac{\underline{h}_3^\circ}{4\underline{h}_3}\left(\frac{\partial_k \underline{h}_3}{\underline{h}_3} + \frac{\partial_k \underline{h}_4}{\underline{h}_4}\right) - \frac{\partial_k \underline{h}_3^\circ}{2\underline{h}_3} \\ &\quad + \frac{h_4}{2h_3}n_k^{**} + \left(\frac{h_4}{h_3}h_3^* - \frac{3}{2}h_4^*\right)\frac{n_k^*}{2h_3} = 0, \end{aligned} \quad (27)$$

$$w_i^* = (\partial_i - w_i)\ln|h_4|, (\partial_k - w_k)w_i = (\partial_i - w_i)w_k, n_i^* = 0, \partial_i n_k = \partial_k n_i, \quad (28)$$

and

$$\begin{aligned} \underline{w}_i^\circ &= 0, \partial_i \underline{w}_k = \partial_k \underline{w}_i, \underline{n}_i^\circ = (\partial_i - \underline{n}_i)\ln|\underline{h}_3|, (\partial_k - \underline{n}_k)\underline{n}_i = (\partial_i - \underline{n}_i)\underline{n}_k, \\ \mathbf{e}_k \omega &= \partial_k \omega - (w_i + \underline{w}_i)\omega^* - (n_i + \underline{n}_i)\omega^\circ = 0. \end{aligned} \quad (29)$$

Proof. It is a consequence of Conclusion 2.1 and Theorems 2.1 and 2.2 for superpositions of ansatz (22) and (23) resulting into (9). If we repeat the computations from Appendix B for geometric data (10) considering that contributions of type $\hat{\Gamma} \cdot \hat{\Gamma}$ are small, we can see that the equations (25)–(27) are derived to be respectively equivalent to sums of (12)–(14) and (17)–(19). The torsionless conditions (28) consist a sum of similar conditions (15) and (20). \square

In general, the solutions defined by a system (24)–(29) can not be transformed into solutions parametrized by an ansatz (22) and/or (23). As we shall prove in Section 3, the general solutions of the such systems of PDE are determined by corresponding sets of generating and integration functions. A solution for (24)–(29) contains a larger set of h - v -generating functions than those with some N-coefficients stated to be zero.

2.2 Splitting of Einstein–Yang–Mills–Higgs equations

In terms of the canonical d-connection $\widehat{\mathbf{D}}$ such nonholonomic interactions are described by PDE,

$$\widehat{\mathbf{R}}_{\beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta} {}^sR = 8\pi G ({}^HT_{\beta\delta} + {}^YM T_{\beta\delta}), \quad (30)$$

$$(\sqrt{|g|})^{-1} D_\mu(\sqrt{|g|}F^{\mu\nu}) = \frac{1}{2}ie[\Phi, D^\nu\Phi], \quad (31)$$

$$(\sqrt{|g|})^{-1} D_\mu(\sqrt{|g|}\Phi) = \lambda(\Phi_{[0]}^2 - \Phi^2)\Phi, \quad (32)$$

where the source is determined by the stress–energy tensor

$${}^HT_{\beta\delta} = Tr[\frac{1}{4}(D_\delta\Phi D_\beta\Phi + D_\beta\Phi D_\delta\Phi) - \frac{1}{4}\mathbf{g}_{\beta\delta}D_\alpha\Phi D^\alpha\Phi] - \mathbf{g}_{\beta\delta}\mathcal{V}(\Phi), \quad (33)$$

$${}^YM T_{\beta\delta} = 2Tr\left(\mathbf{g}^{\mu\nu}F_{\beta\mu}F_{\delta\nu} - \frac{1}{4}\mathbf{g}_{\beta\delta}F_{\mu\nu}F^{\mu\nu}\right). \quad (34)$$

These equations can be derived following a variational principle for a gravitating non-Abelian SU(2) gauge field $\mathbf{A} = \mathbf{A}_\mu \mathbf{e}^\mu$ coupled to a triplet Higgs field Φ as in [15, 16]. For our purposes, the operator $\widehat{\mathbf{D}}$ is used instead of ∇ and all computations are performed with respect N-adapted bases (3) and (4). The nonholonomic interactions of matter fields and constants in (30)–(32) are treated as follows: The gauge field with derivative $D_\mu = \mathbf{e}_\mu + ie[\mathbf{A}_\mu, \cdot]$ in a covariant gravitational background should be changed into $\widehat{D}_\delta = \mathbf{D}_\delta + ie[\mathbf{A}_\delta, \cdot]$. The curvature of vector field \mathbf{A}_δ is

$$F_{\beta\mu} = \mathbf{e}_\beta\mathbf{A}_\mu - \mathbf{e}_\mu\mathbf{A}_\beta + ie[\mathbf{A}_\beta, \mathbf{A}_\mu], \quad (35)$$

where e is the coupling constant, $i^2 = -1$, and $[\cdot, \cdot]$ is used for the commutator. The value $\Phi_{[0]}$ in (32) is the vacuum expectation of the Higgs field which determines the mass ${}^HM = \sqrt{\lambda}\eta$, when λ is the constant of scalar field self-interaction with potential $\mathcal{V}(\Phi) = \frac{1}{4}\lambda Tr(\Phi_{[0]}^2 - \Phi^2)^2$; the gravitational constant G defines the Plank mass $M_{Pl} = 1/\sqrt{G}$ and it is also the mass of gauge boson, ${}^WM = ev$.

For a series of assumptions on nonholonomic constraints, we can decouple the system of PDE (30)–(32) using corresponding splitting for Einstein equations considered in Theorems 2.1, 2.2, or 2.3.

Condition 2.1 *To construct new classes of solutions of off-diagonal EYMH equations we consider that a "prime" solution is known for the system (30)–(32) (given by data for a diagonal d-metric ${}^\circ\mathbf{g} = [{}^\circ g_i(x^1), {}^\circ h_a(x^k),$*

${}^\circ N_i^a = 0]$ and matter fields ${}^\circ A_\mu(x^1)$ and ${}^\circ \Phi(x^1)$, for instance, of type constructed in Ref. [18] (see also Appendix B.5)). We suppose that there are satisfied the following conditions:

1. The "target" d -metric ${}^\eta \mathbf{g}$ with nontrivial N -coefficients, for ${}^\circ \mathbf{g} \rightarrow {}^\eta \mathbf{g}$ is parametrized in a form (9),

$$\begin{aligned} \mathbf{g} &= \eta_i(x^k) {}^\circ g_i(x^1) dx^i \otimes dx^i + \eta_a(x^k, y^a) {}^\circ h_a(x^1, x^2) \mathbf{e}^a \otimes \mathbf{e}^a \\ &= g_i(x^k) dx^i \otimes dx^i + \omega^2(x^k, y^b) h_a(x^k, y^a) \mathbf{e}^a \otimes \mathbf{e}^a, \\ \mathbf{e}^3 &= dy^3 + [w_i + \underline{w}_i] dx^i, \quad \mathbf{e}^4 = dy^3 + [n_i + \underline{n}_i] dx^i. \end{aligned} \quad (36)$$

2. The gauge fields are nonholonomically deformed as

$$A_\mu(x^i, y^3) = {}^\circ A_\mu(x^1) + {}^\eta A_\mu(x^i, y^a), \quad (37)$$

where ${}^\circ A_\mu(x^1)$ is defined by formula (B.8) and ${}^\eta A_\mu(x^i, y^a)$ are any functions for which

$$F_{\beta\mu} = {}^\circ F_{\beta\mu}(x^1) + {}^\eta F_{\beta\mu}(x^i, y^a) = s\sqrt{|g|}\varepsilon_{\beta\mu}, \quad (38)$$

for $s = \text{const}$ and $\varepsilon_{\beta\mu}$ being the absolute antisymmetric tensor. The gauge field curvatures $F_{\beta\mu}$, ${}^\circ F_{\beta\mu}$ and ${}^\eta F_{\beta\mu}$ are computed by introducing (B.8) and (37) into (35). An antisymmetric tensor $F_{\beta\mu}$ (38) satisfies the condition $D_\mu(\sqrt{|g|}F^{\mu\nu}) = 0$, which always give us the possibility to determine ${}^\eta F_{\beta\mu}$, ${}^\eta A_\mu$, for any given ${}^\circ A_\mu$, ${}^\circ F_{\beta\mu}$.

3. The scalar field is nonholonomically modified ${}^\circ \Phi(x^1) \rightarrow \Phi(x^i, y^a) = {}^\Phi \eta(x^i, y^a) {}^\circ \Phi(x^1)$ by a polarization ${}^\Phi \eta$ is such way that

$$D_\mu \Phi = 0 \quad \text{and} \quad \Phi(x^i, y^a) = \pm \Phi_{[0]}. \quad (39)$$

This nonholonomic configuration of the nonlinear scalar field is not trivial even with respect to N -adapted frames $\mathcal{V}(\Phi) = 0$ and ${}^H T_{\beta\delta} = 0$, see formula (33). For ansatz (36), the equations (39) transform into

$$\begin{aligned} (\partial/\partial x^i - A_i)\Phi &= (w_i + \underline{w}_i)\Phi^* + (n_i + \underline{n}_i)\Phi^\circ, \\ (\partial/\partial y^3 - A_3)\Phi &= 0, \quad (\partial/\partial y^4 - A_4)\Phi = 0. \end{aligned} \quad (40)$$

So, a nonolonomically constrained/deformed Higgs Φ field (depending in non-explicit form on two variables because of constraint (39)) modifies indirectly the off-diagonal components of the metric via $w_i + \underline{w}_i$ and $n_i + \underline{n}_i$ and conditions (40) for ${}^\eta A_\mu$.

4. The gauge fields (38) with the potential A_μ (37) modified nonholonomically by Φ subjected to the conditions (39) determine exact solutions of the system (25) and (26) if the spacetime metric is chosen to be in the form (36). The energy-momentum tensor is computed⁷ ${}^{YM}T^\alpha_\beta = -4s^2\delta^\alpha_\beta$. Interacting gauge and Higgs fields, with respect to N -adapted frames, result in an effective cosmological constant ${}^s\lambda = 8\pi s^2$ which should be added to a respective source (A.15).

Now we can formulate a Claim for decoupling PDEs for generic off-diagonal EYM systems:

Claim 2.1 *An ansatz $\mathbf{g} = [\eta_i \circ g_i, \eta_a \circ h_a; w_i, n_i]$ (36) and certain gauge-scalar configurations (A, Φ) subjected to Conditions 2.1 define a decoupling of the EYM system (30)–(32) in a form stated respectively by Theorems 2.1, 2.2, or 2.3 if the sources (A.15) are changed in the form*

$$\Upsilon^\beta_\delta = \text{diag}[\Upsilon_\alpha] \rightarrow \Upsilon^\beta_\delta + {}^{YM}T^\beta_\delta = \text{diag}[\Upsilon_\alpha - 4s^2\delta^\alpha_\beta].$$

Such a statement is a straightforward consequence of above assumptions 1-4 when in N -adapted frames the contributions of matter fields is defined by an effective cosmological constant ${}^s\lambda$. In a particular case, we can state $\Upsilon_\alpha = 0$ if there are not other energy-momentum sources. This statement on nonholonomic gravitational and matter field interactions is formulated as a Claim and not as a Theorem. This follows from our experience that not all EYM systems can be modelled geometrically as nonholonomic Einstein spaces with effective cosmological constant. The solutions of such PDE are defined by very general classes of off-diagonal metrics but the matter field equations and their solutions are substantially constrained nonholonomically and N -adapted to corresponding frames of references in order to encode the data into a nontrivial effective vacuum gravity structure.

The main conclusion of this section is that generic off-diagonal Einstein spaces can be generalized to EYM configurations with effective cosmological constant. Such generalizations preserve the decoupling property of the Einstein equations and impose certain classes of nonholonomic constraints and additional (Pfaff type) first order systems of PDE for matter fields. With respect to coordinate frames, such systems of equations describe very complex, nonlinearly coupled gravitational and gauge-scalar interactions.

⁷such a calculus in coordinate frames is provided in sections 3.2 and 6.51 in Ref. [19]

3 Generic Off-Diagonal Solutions for EYMH Eqs

The goal of this section is to show how the decoupling property of the Einstein equations allows us to integrate such PDE in very general forms depending on properties of coefficients of ansatz for metrics. Some similar theorems for 4-d, 5-d and higher dimension modifications of GR where proven in Refs [10, 11, 9]. In this work, we generalize those results for EYHM systems following Corollary 2.1 and Claim 2.1.

3.1 Generating solutions with weak one Killing symmetry

We prove that the Einstein equations encoding gravitational and YMH interactions and satisfying the conditions of Theorem 2.1 can be integrated in general forms for $h_a^* \neq 0$ and certain special cases with zero and non-zero sources (A.15). In general, such generic off-diagonal metrics are determined by generating functions depending on three/four coordinates. The bulk of known exact solutions with diagonalizable metrics and coefficients depending on two coordinates (in certain special frames of references) can be included as special cases for more general nonholonomic configurations.

3.1.1 (Non) vacuum metrics with $h_a^* \neq 0$

For ansatz (9) with data $\omega = 1, \underline{h}_3 = 1, \underline{w}_i = 0$ and $\underline{n}_i = 0$ for (10), when $h_a^* \neq 0$, and the condition that the source

$$\Upsilon^\beta_\delta = \text{diag}[\Upsilon_\alpha : \Upsilon^1_1 = \Upsilon^2_2 = \Upsilon(x^k, y^3) - 4s^2; \Upsilon^3_3 = \Upsilon^4_4 = {}^v\Upsilon(x^k) - 4s^2], \quad (41)$$

is not zero, the solutions of Einstein eqs can be constructed following

Theorem 3.1 *The EYHM equations of type (11)–(14) with source (41) can be integrated in general forms by metrics*

$$\begin{aligned} \mathbf{g} &= \epsilon_i e^{\psi(x^k)} dx^i \otimes dx^i + h_3(x^k, y^3) \mathbf{e}^3 \otimes \mathbf{e}^3 + h_4(x^k, y^3) \underline{h}_4(x^k, y^4) \mathbf{e}^4 \otimes \mathbf{e}^4, \\ \mathbf{e}^3 &= dy^3 + w_i(x^k, y^3) dx^i, \quad \mathbf{e}^4 = dy^4 + n_i(x^k) dx^i, \end{aligned} \quad (42)$$

with coefficients determined by generating functions $\psi(x^k), \phi(x^k, y^3), \phi^* \neq 0, n_i(x^k)$ and $\underline{h}_4(x^k, y^4)$, and integration functions ${}^0\phi(x^k)$ following recur-

rent formulas and conditions

$$\epsilon_1 \psi^{\bullet\bullet} + \epsilon_2 \psi'' = 2 [\ ^v\Upsilon - 4s^2]; \quad (43)$$

$$h_4 = \pm \frac{1}{4} \int |\Upsilon - 4s^2|^{-1} (e^{2\phi})^* dy^3, \text{ or} \quad (44)$$

$$= \pm \frac{1}{4(\Lambda - 4s^2)} e^{2[\phi - {}^0\phi]}, \text{ if } \Upsilon = \Lambda = \text{const} \neq 4s^2;$$

$$h_3 = \pm \left[\left(\sqrt{|h_4|} \right)^* \right]^2 e^{-2\phi} = \frac{\phi^*}{2|\Upsilon - 4s^2|} (\ln |h_4|)^* \text{ or} \quad (45)$$

$$= \pm \frac{(\phi^*)^2}{4(\Lambda - 4s^2)}, \text{ if } \Upsilon = \Lambda = \text{const} \neq 4s^2;$$

$$w_i = -\partial_i \phi / \phi^*, \quad (46)$$

where constraints

$$w_i^* = (\partial_i - w_i) \ln |h_4|, \partial_k w_i = \partial_i w_k, \quad (47)$$

$$n_k \underline{h}_4^\circ = \partial_k \underline{h}_4, \partial_i n_k = \partial_k n_i.$$

must be imposed in order to satisfy the zero torsion conditions (15); we should take respective values $\epsilon_i = \pm 1$ and \pm in (44) and/or (45) if we want to fix a necessary spacetime signature.

Proof. We sketch a proof which transforms into similar ones in [10, 11, 9] if $\underline{h}_4 = 1$.

- A horizontal metric $g_i(x^2)$ is for 2-d and can be always represented in a conformally flat form $\epsilon_i e^{\psi(x^k)} dx^i \otimes dx^i$. For such a h-metric, the equation (11) is a 2-d Laplace/wave equation (43) which can be solved exactly if a source ${}^v\Upsilon(x^k) - 4s^2$ is prescribed from certain physical conditions.
- If $h_4^* \neq 0$, we can define nontrivial functions

$$\phi = \ln \left| \frac{h_4^*}{\sqrt{|h_3 h_4|}} \right|, \gamma := \left(\ln \frac{|h_4|^{3/2}}{|h_3|} \right)^*, \alpha_i = h_4^* \partial_i \phi, \beta = h_4^* \phi^* \quad (48)$$

for a function $\phi(x^k, y^3)$. If $\phi^* \neq 0$, we can write respectively the equations (12)–(14) in the forms,

$$\phi^* h_4^* = 2h_3 h_4 [\Upsilon - 4s^2]. \quad (49)$$

$$\beta w_i + \alpha_i = 0, \quad (50)$$

$$n_i^{**} + \gamma n_i^* = 0.$$

For the last equation, we must take any trivial solution given by functions $n_i(x^k)$ satisfying the conditions $\partial_i n_j = \partial_j n_i$ in order to solve the constraints (15). Using coefficients (48) with $\alpha_i \neq 0$ and $\beta \neq 0$, we can always express w_i via derivatives of ϕ , i.e. in the form (46). We can chose any $\phi(x^k, y^3)$ with $\phi^* \neq 0$ as a generating function and express h_4 and, after that, h_3 as some integrals/derivatives of functions depending on ϕ and source $\Upsilon(x^k, y^3) - 4s^2$, see corresponding formulas (44) and (45). The integrals can be computed in a general explicit form if $\Upsilon(x^k, y^3) = \Lambda = \text{const} \neq 4s^2$, when other possible matter field (additionally to the considered YMH ones) interactions are approximated by an energy-momentum tensor as a cosmological constant,

$$\Upsilon^\beta_\delta = \delta^\beta_\delta \Upsilon = \delta^\beta_\delta \Lambda. \quad (51)$$

- A possible dependence on y^4 is present in function h_4 which must satisfy conditions of type (B.6) in order to be compatible with (15). It is not possible to write in explicit form the solutions for the zero torsion condition if the source Υ^β_δ is parametrized by arbitrary functions. Nevertheless, if Υ^β_δ is of type (51), we get $h_4 \sim e^{2\phi}$ and $w_i \sim \partial_i \phi / \phi^*$ positively solve the constraints

$$w_i^* = (\partial_i - w_i) \ln |h_4| \text{ and } \partial_i w_j = \partial_j w_i, \quad (52)$$

transformed into $\phi^{**} \partial_i \phi - \phi^* (\partial_i \phi)^* = 0$. By straightforward computation, we can check that (15) are satisfied by (47) when $n_i^* = 0$ and w_i is determined by (46). \square

The solutions constructed in Theorem 3.1, and those which can be derived following Corollary 2.1 are very general ones and contain as particular cases (pehaps) all known exact solutions for (non) holonomic Einstein spaces with Killing symmetries. They also can be generalized to include arbitrary finite sets of parameters as it is proven in Ref. [9].

Remark 3.1 (-Example) *Introducing $H = -\ln |h_4|$, the sistem of equations (49), (50) and (52) can be written in the form*

$$h_3 = -\phi^* H^* / 2\Upsilon_2, \quad (53)$$

$$\phi^* w_i + \partial_i \phi = 0, \quad (54)$$

$$w_i^* + \partial_i H - w_i H^* = 0. \quad (55)$$

Considering $H(x^k, y^3)$ as a generating function, we can integrate (55) in explicit form using parametrizations $w_i = {}_1w_i(x^k, y^3) \times {}_2w_i(x^k, y^3)$ (in this formula, we do not consider summation on i). The function $\phi(x^k, y^3)$ can be any one for which the system of first order PDE (54) has nontrivial solutions for any found w_i . So, we can define h_3 for any prescribed source Υ_2 in (53). We conclude that the LC-conditions (15) (relating w_i to h_4 , which does not allow a "complete" decoupling) can be solved also in very general forms by further fixing of parametrizations, classes of functions and boundary conditions for H, ϕ and Υ_2 . An explicit example of exact solutions can be derived by taking derivative on $*$ for (54) and using a subclass of functions ϕ when $(\partial_i \phi)^* = \partial_i(\phi^*)$. We obtain $w_i \frac{\phi^{**}}{\phi^*} + w_i^* + \frac{\partial_i \phi^*}{\phi^*} = 0$. This system is equivalent/compatible to (55) if, for instance, for $\Phi := \ln |\phi^*|$,

$$H^* = -\Phi^*, \quad \partial_i H = \partial_i \Phi.$$

Such equations are satisfied by any functions $H(\mp \varsigma)$ and $\Phi(\pm \varsigma)$, where $\pm \varsigma := \pm v - x^1 - x^2$.

3.1.2 Effective vacuum EYMH configurations

We can consider a subclass of generic off-diagonal EYMH interactions which can be encoded as effective Einstein manifolds with nontrivial cosmological constant $\Lambda = 4s^2$. In general, such classes of solutions depend parametrically on $\Lambda - 4s^2$ and do not have a smooth limit from non-vacuum to vacuum models.

Corollary 3.1 *The effective vacuum solutions for the EYHM equations with ansatz for metrics of type (42) with vanishing source (41) are parametrized in the form*

$$\begin{aligned} \mathbf{g} &= \epsilon_i e^{\psi(x^k)} dx^i \otimes dx^i + h_3(x^k, y^3) \mathbf{e}^3 \otimes \mathbf{e}^3 + h_4(x^k, y^3) \underline{h}_4(x^k, y^4) \mathbf{e}^4 \otimes \mathbf{e}^4, \\ \mathbf{e}^3 &= dy^3 + w_i(x^k, y^3) dx^i, \quad \mathbf{e}^4 = dy^4 + n_i(x^k) dx^i, \end{aligned} \quad (56)$$

where coefficients are defined by solutions of the system

$$\ddot{\psi} + \psi'' = 0, \quad (57)$$

$$\phi^* h_4^* = 0, \quad (58)$$

$$\beta w_i + \alpha_i = 0, \quad (59)$$

where coefficients are computed following formulas (48) for nonzero ϕ^* and h_4^* with possible further zero limit; such coefficients and h_3 and \underline{h}_4 are subjected additionally to the zero-torsion conditions (47).

Proof. Considering the system of equations (11)–(14) with zero right sides, we obtain respectively the equations (57)–(59). For positive signatures on h-subspace and equation (57), we can take $\psi = 0$, or consider a trivial 2-d wave equation if one of coordinates x^k is timelike. There are two possibilities to satisfy the condition (58). The first one is to consider that $h_4 = h_4(x^k)$, i.e. h_4^* which states that the equation (58) has solutions with zero source for arbitrary function $h_3(x^k, y^3)$ and arbitrary N-coefficients $w_i(x^k, y^3)$ as follows from (48). For such vacuum configurations, the functions h_3 and w_i can be taken as generation ones which should be constrained only by the conditions (47). Equations of type (52) constrain substantially the class of admissible w_i if h_4 depends only on x^k . Nevertheless, h_3 can be an arbitrary one generating solutions which can be extended for nontrivial sources $\underline{\mathfrak{T}}$ and systems (16)–(20) and/or (24)–(29).

A different class of solutions can be generated if we state, after corresponding coordinate transforms, $\phi = \ln |h_4^*/\sqrt{|h_3 h_4|}| = {}^0\phi = \text{const}$, $\phi^* = 0$. For such configurations, we can consider $h_4^* \neq 0$, and solve (58) as

$$\sqrt{|h_3|} = {}^0h(\sqrt{|h_4|})^*, \quad (60)$$

for ${}^0h = \text{const} \neq 0$. Such v-metrics are generated by any $f(x^i, y^3)$, $f^* \neq 0$, when

$$h_4 = -f^2(x^i, y^3) \text{ and } h_3 = ({}^0h)^2 [f^*(x^i, y^3)]^2, \quad (61)$$

where the sines are such way fixed that for $N_i^a \rightarrow 0$ we obtain diagonal metrics with signature $(+, +, +, -)$. The coefficients $\alpha_i = \beta = 0$ in (59) and $w_i(x^k, y^3)$ can be any functions solving (47). This is equivalent to

$$\begin{aligned} w_i^* &= 2\partial_i \ln |f| - 2w_i(\ln |f|)^*, \\ \partial_k w_i - \partial_i w_k &= 2(w_k \partial_i - w_i \partial_k) \ln |f|, \end{aligned} \quad (62)$$

for any $n_i(x^k)$ when $\partial_i n_k = \partial_k n_i$. Constraints of type $n_k \underline{h}_4^\circ = \partial_k \underline{h}_4$ (B.6) have to be imposed for a nontrivial multiple \underline{h}_4 . \square

Using Corollary 2.1, the ansatz (56) can be "dualized" to generate effective vacuum solutions with weak Killing symmetry on $\partial/\partial y^3$. Finally, we note that the signature of the generic off-diagonal metrics generated in this subsection depend on the fact which coordinate x^1, x^2, y^3 or y^4 is chosen to be a timelike one.

3.2 Non-Killing EYMH configurations

The Theorem 2.2 can be applied for constructing non-vacuum and effective vacuum solutions of the EYMH equations depending on all coordinates

without explicit Killing symmetries.

3.2.1 Non-vacuum off-diagonal solutions

We can generate such YMH Einstein manifolds following

Corollary 3.2 *An ansatz of type (22) with d-metric*

$$\begin{aligned} \mathbf{g} &= \epsilon_i e^\psi dx^i \otimes dx^i + \omega^2 [\pm \frac{(\phi^*)^2}{4(\Lambda - 4s^2)} \mathbf{e}^3 \otimes \mathbf{e}^3 \pm \frac{1}{4(\Lambda - 4s^2)} e^{2[\phi - {}^0\phi]} \underline{h}_4 \mathbf{e}^4 \otimes \mathbf{e}^4], \\ \mathbf{e}^3 &= dy^3 - (\partial_i \phi / \phi^*) dx^i, \quad \mathbf{e}^4 = dy^4 + n_i(x^k) dx^i, \end{aligned}$$

where the coefficients are subjected to conditions (43)–(47) and $\partial_k \omega + (\partial_i \phi / \phi^*) \omega^* - n_i \omega^\circ = 0$, defines solutions of the Einstein equations $R_{\alpha\beta} = (\Lambda - 4s^2) g_{\alpha\beta}$ with nonholonomic interactions of YMH fields encoded effectively in the vacuum structure of GR with nontrivial cosmological constant, $\Lambda - 4s^2 \neq 0$.

Proof. It is a consequence of Theorems 2.2 and Corollary 3.1. \square

In a similar form, we can generate solutions of type (23) when the conformal factor is a solution of $\partial_k \omega - \underline{w}_i(x^k) \omega^* + (\partial_i \phi / \phi^\circ) \omega^\circ = 0$ with respective "dual" generating functions ω and ϕ when the data (43)–(47) are re-defined for solutions with weak Killing symmetry on $\partial/\partial y^3$.

3.2.2 Effective vacuum off-diagonal solutions

Vacuum Einstein spaces encoding nonholonomic interactions of EYMH fields can be constructed using

Corollary 3.3 *An ansatz of type (22) with d-metric*

$$\begin{aligned} \mathbf{g} &= \epsilon_i e^{\psi(x^k)} dx^i \otimes dx^i + \omega^2(x^k, y^a) [({}^0h)^2 [f^*(x^i, y^3)]^2 \mathbf{e}^3 \otimes \mathbf{e}^3 \\ &\quad - f^2(x^i, y^3) \underline{h}_4(x^k, y^4) \mathbf{e}^4 \otimes \mathbf{e}^4], \\ \mathbf{e}^3 &= dy^3 + w_i(x^k, y^3) dx^i, \quad \mathbf{e}^4 = dy^4 + n_i(x^k) dx^i, \end{aligned}$$

where the coefficients are subjected to conditions (60)–(62), (47) and $\partial_k \omega - w_i \omega^* - n_i \omega^\circ = 0$, define generic off-diagonal solutions of $R_{\alpha\beta} = 0$.

Proof. It is a consequence of Theorem 2.2 and Corollary 3.1. \square

Solutions of type (23) can be defined if the conformal factor is a solution of $\partial_k \omega - \underline{w}_i(x^k) \omega^* + \underline{n}_i(x^k, y^4) \omega^\circ = 0$ with respective "dual" generating

functions $\omega(x^k, y^a)$ and $\phi(x^k, y^4)$ when the data (60)–(62) and (47) are re-defined for ansatz with weak Killing symmetry on $\partial/\partial y^3$.

Summarizing the results of this section, we formulate

Claim 3.1 *All generic off-diagonal non-vacuum and effective vacuum solutions of the EYM equations determined respectively by Theorem 3.1 and Corollaries 3.1, 3.2 and 3.3 can be generalized to metrics of type (9). For such constructions, there are used nonlinear superposition of metrics and their "duals" on v -coordinates in order to define non-Killing solutions of respective systems of PDEs from Theorems 2.2 and/or 2.3*

The statements of this Claim are formulated following our experience on constructing generic off-diagonal vacuum and non-vacuum solutions. For such nonlinear systems, it is not possible to formulate certain general uniqueness and exhaustive criteria. Almost sure, not all solutions of EYM equations can be constructed in such forms, or related to any sets of prescribed solutions via "nondegenerate" nonholonomic deformations. The length of this paper does not allow us to present all technical details and general formulas for coefficients for ansatz for d-metrics⁸ which can be constructed following this Claim. Explicit examples supporting our approach are given in sections 5 and 6.

4 The Cauchy Problem and Decoupling of EYM Equations

The gravitational interactions of EYM systems studied in this work are described by off-diagonal solutions of

$$R_{\alpha\beta} = (\Lambda - 4s^2)g_{\alpha\beta}, \quad (63)$$

which can be found in very general forms with respect to N-adapted frames for certain nonintegrable spacetime $2 + 2$ splitting of type **N** (2). An effective cosmological constant $(\Lambda - 4s^2)$ encodes a gravitational "vacuum" cosmological constant Λ in GR and s^2 is induced by nonholonomic dynamics of YM fields. Pseudo-Riemannian manifolds with metrics $g_{\alpha\beta}$ adapted to chosen non-integrable distribution with $2 + 2$ splitting and satisfying (63) are called *nonholonomic Einstein manifolds*. Hereafter we shall refer to such

⁸such formulas are, for instance, of type (43)–(47), with functions $h_3(., y^3)$ and $h_4(., y^3)$ and further "dualization" $h_3 \rightarrow \underline{h}_4(., y^4)$ and $h_4 \rightarrow \underline{h}_3(., y^4)$ with corresponding re-definition of N-connection coefficients

systems of PDEs as *nonholonomic vacuum spacetimes*, regardless of whether or not an (effective) cosmological constant vanishes or can be "polarized" by gravitational and/or matter field interactions into some N-adapted diagonal sources admitting formal integration of gravitational field equations.

The equations (63), and their N-adapted equivalents (A.12)–(A.13), constitute a second-order system quasi-linear PDE for the coefficients of space-time metric $\mathbf{g} = \{g_{\alpha\beta}\}$. This means that given a manifold \mathbf{V} of necessary smooth class such a quasi-linear system is linear in the second derivatives of the metric and quadratic in the first derivatives $\partial_\gamma g_{\alpha\beta}$ (the coefficients of such PDE are rational functions of $g_{\alpha\beta}$). To be able to decouple and formally integrate such systems is necessary to consider special classes of nonholonomic frames and constraints. For GR, this type of equations do not fall in any of the standard cases of hyperbolic, parabolic, or elliptic systems which typically lead to unique solutions. It is important to formulate criteria when such general solutions would be unique ones with a topology and differential structure determined by some initial data. How the diffeomorphism, or coordinate, invariance and arbitrary frame transforms (principle of relativity) would be taken into account for N-splitting?

In mathematical relativity [1, 3, 4, 5, 6, 7, 8], it was proven a fundamental result (due to Choquet–Bruhat, 1952) that there exists a set of hyperbolic equations underlying (63). The goal of this section is to study the evolution (Cauchy) problem for the system (A.12)–(A.13) in N-adapted form and preserving the decoupling property.

4.1 The local N-adapted evolution problem

In the evolutionary approach, the topology of spacetime manifold is chosen in the form $\mathbf{V} = \mathbb{R} \times {}^3V$, where 3V is a 3-d manifold carrying initial data. It should be noted here that there is no a priori known natural time-coordinate even we may fix a signature for metric and choose certain coordinates to be time or space like ones.

Definition 4.1 *A set of coordinates $\{\hat{u}^\mu = (\hat{x}^i, \hat{y}^a)\}$ is canonically N-harmonic, i.e. it both harmonic and adapted to a splitting \mathbf{N} (2), if each of the functions \hat{u}^μ satisfies the wave equation*

$$\hat{\square} \hat{u}^\mu = 0, \quad (64)$$

where the canonical d’Alembert operator $\hat{\square} := \hat{\mathbf{D}}_\alpha \hat{\mathbf{D}}^\alpha$ acts on a scalar

$f(x, y)$ in the form

$$\begin{aligned}\hat{\square}f &:= (\sqrt{|\mathbf{g}_{\alpha\beta}|})^{-1}\mathbf{e}_\mu \left(\sqrt{|\mathbf{g}_{\alpha\beta}|}\mathbf{g}^{\mu\nu}\mathbf{e}_\nu f \right) \\ &= (\sqrt{|g_{kl}|})^{-1}\mathbf{e}_i \left(\sqrt{|g_{kl}|}g^{ij}\mathbf{e}_j f \right) + (\sqrt{|g_{cd}|})^{-1}\partial_a \left(\sqrt{|g_{kl}|}g^{ab}e_b f \right),\end{aligned}$$

for a d -metric $\mathbf{g}_{\alpha\beta} = (g_{ij}, g_{ab})$ (7) defined with respect to N -adapted (co) frames (3)–(4) and canonical d -connection $\hat{\mathbf{D}}_\alpha$ (A.3).

We can say that four such coordinates $\hat{u}^\mu = (\hat{x}^i, \hat{y}^a)$ are N -adapted wave-coordinates.

Lemma 4.1 *In canonical N -harmonic coordinates, the Einstein equations (63) re-defined in canonical d -connection variables (A.12) can be written equivalently*

$$\begin{aligned}\hat{\mathbf{E}}^{\alpha\beta} &= \hat{\square}\mathbf{g}^{\alpha\beta} - \mathbf{g}^{\tau\theta} \left[(\mathbf{g}^{\alpha\mu}\hat{\Gamma}_{\mu\nu}^\beta + \mathbf{g}^{\alpha\mu}\hat{\Gamma}_{\mu\nu}^\beta)\hat{\Gamma}_{\tau\theta}^\nu + 2\mathbf{g}^{\gamma\mu}\hat{\Gamma}_{\mu\theta}^\alpha\hat{\Gamma}_{\tau\gamma}^\beta \right] \\ &\quad - 2(\Lambda - 4s^2)\mathbf{g}^{\alpha\beta} = 0,\end{aligned}\tag{65}$$

i.e. such PDE for $\mathbf{g}^{\alpha\beta}$ (using algebraic transforms, for $\mathbf{g}_{\alpha\beta}$) form a system of second-order quasi-linear N -adapted wave-type equations.

Proof. It is a standard computation with respect to N -adapted frames by using formulas (A.3), (A.9) and (A.7). If the zero torsion conditions (A.13) are imposed, we get well known results from GR but (in our case) adapted to $2 + 2$ non-integrable splitting. \square

This Lemma allows us to apply the standard theory of hyperbolic PDE (see, for instance, [20]). Let us denote by H_{loc}^k the Sobolev spaces of functions which are in $L^2(K)$ for any compact set K when their distributional derivatives are considered up to an integer order k also in $L^2(K)$. We shall also use N -adapted wave coordinates with additional formal $3 + 1$ splitting, for instance, in a form $\hat{u}^\mu = ({}^t\hat{u}, \hat{u}^{\bar{i}})$, where ${}^t\hat{u}$ is used for the timelike coordinate and $\hat{u}^{\bar{i}}$ are for 3 spacelike coordinates. "Hats" can be eliminated if such a splitting is considered for arbitrary local coordinates. Standard results from the theory of PDE give rise to this

Theorem 4.1 *The field equations (A.12) for nonholonomic Einstein manifolds have a unique solution $\mathbf{g}^{\alpha\beta}$ defined by PDE (65) stated on an open neighborhood $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^3$ of $\mathcal{O} \subset \{0\} \times \mathbb{R}^3$ with any initial data*

$$\mathbf{g}^{\alpha\beta}(0, \hat{u}^{\bar{i}}) \in H_{loc}^{k+1} \quad \text{and} \quad \frac{\partial \mathbf{g}^{\alpha\beta}(0, \hat{u}^{\bar{i}})}{\partial ({}^t\hat{u})} \in H_{loc}^{k+1}, k > 3/2. \tag{66}$$

The set \mathcal{U} can be chosen in a form that $(\mathcal{U}, \mathbf{g}^{\alpha\beta})$ is globally hyperbolic with Cauchy surface \mathcal{O} .⁹

There is no reason that a solution constructed using the anholonomic deformation method as we considered in section 3 will satisfy the wave conditions (64) even we may give certain initial data for an equation (65). In order to establish an hyperbolic (and evolutionary) form of the such non-holonomic vacuum gravitational equations we should re-define the Choquet-Bruhat problem (see details and references in [1, 3, 4, 6, 8]) with respect to N-adapted frames. Using a 3+1 splitting of N-adapted coordinates, we write $\mathbf{g}^{\alpha\beta} = (\mathbf{g}^{t\beta}, \mathbf{g}^{\bar{i}\beta})$ and $\mathbf{e}_\alpha = (\mathbf{e}_t, \mathbf{e}_{\bar{i}})$ and consider the d-vector field $n^\mu(x, y)$ of unit timelike normals to the hypersurface $\{ {}^t\hat{u} = 0 \}$.¹⁰ There is no loss of generality if we assume that on such a hypersurface we have

$$\mathbf{g}^{tt} = -1 \text{ and } \mathbf{g}^{t\bar{i}} = 0. \quad (67)$$

We can state such conditions via additional N-adapted frame/coordinate transform for any d-metric (7) with prescribed signature. It is also possible to re-define the generating functions for any class of off-diagonal solutions with decoupling of Einstein equations in order to satisfy (67).

Another necessary condition for the vanishing of a $\hat{\square}u^\mu$ is that this value is stated zero at ${}^t\hat{u} = 0$. This allows us to express in N-adapted form the time like derivatives through N-elongated space type ones,

$$\mathbf{e}_t \left(\sqrt{|\mathbf{g}_{\alpha\beta}|} \mathbf{g}^{t\bar{i}} \right) = -\mathbf{e}_{\bar{i}} \left(\sqrt{|\mathbf{g}_{\alpha\beta}|} \mathbf{g}^{t\bar{i}} \right) \quad (68)$$

with N-elongated operators. So, the initial data from Theorem 4.1 can not be fixed in arbitrary form if we want to establish a hyperbolic (evolutionary) character for nonholonomic vacuum Einstein equations, i.e. to satisfy both systems (65) and (64). Really, the last system of first order PDE allows us to compute the time-like derivatives $\partial \mathbf{g}^{t\bar{i}}(0, \hat{u}^{\bar{i}}) / \partial {}^t\hat{u} |_{\{ {}^t\hat{u}=0 \}}$ if $\mathbf{g}_{\bar{i}\bar{j}} |_{\{ {}^t\hat{u}=0 \}}$ and $\partial \mathbf{g}_{\bar{i}\bar{j}} / \partial {}^t\hat{u} |_{\{ {}^t\hat{u}=0 \}}$ have been defined. We conclude that the essential data for formulating a N-adapted evolution problem should be formulated for a 3-d space d-metric

$$^{[3]}\mathbf{g} := \mathbf{g}_{\bar{i}\bar{j}}(x, y) \mathbf{e}^{\bar{i}} \otimes \mathbf{e}^{\bar{j}}, \quad (69)$$

⁹Choosing corresponding classes of nonholonomic distributions \mathbf{N} (2), we can relax the conditions of differentiability as in Refs. [21, 22] (we omit such constructions in this work).

¹⁰We do not use labels for coordinates like 0, 1, 2, 3 because the decoupling property of the Einstein equations and general solutions can be proven for arbitrary signature, for instance $(-+++), (++-+)$ etc and for any set of local coordinates u^α with $\alpha = 1, 2, 3, 4$.

where $\bar{\mathbf{e}}^{\bar{j}}$ are N-elongated differentials of type (4), and its N-elongated (3) time-like derivatives.

Using the Theorem 4.1 and above presented considerations, we get the proof of

Theorem 4.2 *If the initial data (66) satisfy the conditions (67) and (68) and the so-called Einstein N-adapted constraint equations,*

$$\left(\hat{\mathbf{E}}_{\alpha\beta} + \Lambda \mathbf{g}_{\alpha\beta}\right) n^\alpha = 0, \quad (70)$$

are computed for zero distortion in (A.1), then the d-metric stated by Theorem 4.1 defines solutions of the nonholonomic vacuum equations (63) and/or (A.12)–(A.13).

This theorem gives us the possibility to state the Cauchy data for decoupled EYMH systems and their generic off-diagonal solutions in order to generate N-adapted evolutions.

4.2 On initial data sets and global nonholonomic evolution

We adopt this convention for spacetime nonholonomic manifolds $\mathbf{V} = \mathbb{R} \times {}^3V$, where 3V is a N-adapted 3-d manifold, when a Whitney sum $T {}^3V = h {}^3V \oplus v {}^3V$ is stated by a space like nonholonomic distribution ${}^3\mathbf{N}$ of type (2) and there is an embedding $e : {}^3V \rightarrow \mathbf{V}$.

Using the d-metric ${}^{[3]}\mathbf{g}$ (69), we can define the second fundamental form \mathbf{K} of a spacelike hypersurface 3V in \mathbf{V} , $\hat{\mathbf{K}}(\mathbf{X}, \mathbf{Y}) := \mathbf{g}(\hat{\mathbf{D}}_{\mathbf{X}}\mathbf{n}, \mathbf{Y})$, $\forall \mathbf{X} \in T {}^3V$. The unity d-vector $\mathbf{n} = n^\alpha \mathbf{e}_\alpha = n_{\bar{i}} \bar{\mathbf{e}}^{\bar{i}} = n_t \mathbf{e}^t = \left(\sqrt{|g^{tt}|}\right)^{-1} \mathbf{e}^t$, with normalization $\mathbf{g}(\mathbf{n}, \mathbf{n}) = \mathbf{g}^{\alpha\beta} n_\alpha n_\beta = \mathbf{g}^{tt} (n_t)^2 = -1$, is time-future and normal to 3V . The value $\hat{\mathbf{K}} = \{\hat{K}_{\bar{i}\bar{j}}\}$ is the extrinsic canonical curvature d-tensor of 3V . Imposing the zero torsion conditions (A.13), $\hat{\mathbf{K}} \rightarrow \{K_{\bar{i}\bar{j}}\}$, where $K_{\bar{i}\bar{j}} = -\frac{1}{2} \mathbf{g}^{t\alpha} \left(\mathbf{e}_{\bar{j}} \mathbf{g}_{\alpha\bar{i}} + \mathbf{e}_{\bar{i}} \mathbf{g}_{\alpha\bar{j}} - \mathbf{e}_\alpha \mathbf{g}_{\bar{i}\bar{j}} \right) n_t$ are components computed in standard form using the Levi-Civita connection, but with respect to N-adapted frames. We can invert the last formula and write $\partial_t \mathbf{g}_{\bar{i}\bar{j}} = 2(\mathbf{g}^{tt} n_t)^{-1} K_{\bar{i}\bar{j}} + \{\text{terms determined by } \mathbf{g}_{\alpha\beta} \text{ and their space-derivatives}\}$. Such formulas show that $K_{\bar{i}\bar{j}}$ and $\partial_t \mathbf{g}_{\bar{i}\bar{j}}$ are geometric counterparts on hypersurfaces $\{ {}^t\hat{u} = 0 \}$.

For the canonical d-connection $\hat{\mathbf{D}}_\alpha = \left(\hat{D}_t, \hat{\mathbf{D}}_{\bar{i}} \right)$ and d-metric ${}^{[3]}\mathbf{g}$ induced on a spacelike hypersurface in a Lorentzian nonholonomic manifold \mathbf{V} , we can derive a N-adapted variant of Gauss-Codazzi equations

$${}^{[3]}\hat{R}_{\bar{j}\bar{k}\bar{l}}^{\bar{i}} = \hat{\mathbf{R}}_{\bar{j}\bar{k}\bar{l}}^{\bar{i}} + \hat{K}_{\bar{l}}^{\bar{i}} \hat{K}_{\bar{j}\bar{k}} - \hat{K}_{\bar{k}}^{\bar{i}} \hat{K}_{\bar{j}\bar{l}}, \quad \hat{\mathbf{D}}_{\bar{i}} \hat{K}_{\bar{j}\bar{k}} - \hat{\mathbf{D}}_{\bar{j}} \hat{K}_{\bar{i}\bar{k}} = \hat{\mathbf{R}}_{\bar{j}\bar{k}\bar{l}}^{\bar{i}} n^{\bar{l}}.$$

In these formulas, ${}^{[3]}\widehat{R}^i_{\overline{jkl}}$ is the canonical curvature d-tensor of ${}^{[3]}\mathbf{g}$, $\widehat{\mathbf{R}}^i_{\overline{jkl}}$ is computed following formulas (A.6) as the spacetime canonical d-curvature tensor, \mathbf{n} is the timelike normal to hypersurface 3V when the local N-adapted coordinate system is such way chosen that d-vectors $\mathbf{e}_{\overline{j}}$ are tangent to 3V . Contracting indices, introducing divergence operator \widehat{div} determined by N-elongated partial derivatives (3), trace operator tr and absolute differential d , we derive from above equations and (A.12) this system of equations:

$$\begin{aligned}\widehat{div}\widehat{K} - d(tr\widehat{K}) &= 8\pi\widehat{J}, \text{ momentum constraint; (71)} \\ {}^s_{[3]}\widehat{R} - 2\Lambda - {}^{[3]}\mathbf{g}^{\overline{i}\overline{j}}\widehat{K}^i_{\overline{l}}\widehat{K}^l_{\overline{j}} + (tr\widehat{K})^2 &= 16\pi\widehat{\rho}, \text{ Hamiltonian constraint;} \\ \mathcal{C}(\widehat{\mathcal{F}}, {}^{[3]}\mathbf{g}) &= 0, \text{ Einstein constraint eqs;}\end{aligned}$$

where ${}^s_{[3]}\widehat{R}$ is computed as the scalar (A.8) but for ${}^{[3]}\mathbf{g}$. In a general context, we consider that 3V is embedded in a nonholonomic spacetime \mathbf{V} with induced data $\left({}^3V, {}^{[3]}\mathbf{g}, \widehat{\mathbf{K}}, \widehat{\mathcal{F}}\right)$, we have $\widehat{J} := -(\mathbf{n}, \cdot)$ and $\widehat{\rho} := (\mathbf{n}, \mathbf{n})$. The term $\mathcal{C}(\widehat{\mathcal{F}}, {}^{[3]}\mathbf{g})$ denotes the set of additional constraints resulting from the non-gravitational part, including nonholonomic distributions \mathbf{N} (2). If in such a set there are included the zero torsion conditions (A.13), we can omit "hats" on geometric/physical objects if they are written in "not" N-adapted frames of reference. The equations (71) form an undetermined system of PDE. For 3-d, there are locally four equations for twelve unknown values give by the components of d-tensors ${}^{[3]}\mathbf{g}$ and $\widehat{\mathbf{K}}$. Using the conformal method with the Levi-Civita connection, see details and references in Ref. [23], we can study the existence and uniqueness of solutions to such systems.

Above considerations motivate

Definition 4.2 *A canonical vacuum initial N-adapted data set is defined by a triple $\left({}^3V, {}^{[3]}\mathbf{g}, \widehat{\mathbf{K}}\right)$, where $\left({}^{[3]}\mathbf{g}, \widehat{\mathbf{K}}\right)$ are defined as a solution of (71).*

If the conditions (A.13) are imposed, the data $\left({}^{[3]}\mathbf{g}, \widehat{\mathbf{K}}\right)$ are equivalent to similar ones $\left({}^{[3]}\mathbf{g}, \mathbf{K}\right)$ with \mathbf{K} computed for the Levi-Civita connection. Covering 3V by coordinate neighborhoods \mathcal{O}_u of points $u \in {}^3V$, we can use Theorem 4.1 to construct globally hyperbolic N-adapted developments $(\mathcal{U}_u, \mathbf{g}_u)$ of an initial data set $\left(\mathcal{U}_u \subset {}^3V, {}^{[3]}\mathbf{g}, \widehat{\mathbf{K}}\right)$ as in above Definition. The d-metrics generated in such forms will coincide after performing suitable N-adapted frame/coordinated transforms on charts covering such a spacetime wherever such solutions are defined. We can patch all data $(\mathcal{U}_u, \mathbf{g}_u)$ together to a globally hyperbolic Lorenzian nonholonomic spacetime containing a Cauchy surface 3V . This provides a proof of

Theorem 4.3 *Any N -adapted initial data $({}^3V, {}^{[3]}\mathbf{g}, \widehat{\mathbf{K}})$ of differentiability class $H^{s+1} \times H^s$, $s > 3/2$, admits a globally hyperbolic, N -adapted and unique development (in the sense of Theorem 4.1 and above considered assumptions and proofs).*

For the EYM systems defined by assumptions in Condition 2.1, we encode the data on such gravitational and gauge–scalar interactions into the nonholonomic structure of certain effective Einstein manifolds.

5 Anholonomic YM Deformations of Black Holes

Gauge–Higgs nonholonomic interactions can be parametrized in such forms that they define off–diagonal deformations (for instance, of rotoid type) of Schwarzschild black holes. In this section, we study such EYM configurations when $\Lambda + {}^s\lambda = 0$. Nonholonomic deformations can be derived from any “prime” data $({}^\circ\mathbf{g}, {}^\circ\mathbf{A}_\mu, {}^\circ\Phi)$ stating, for instance a diagonal cosmological monopole and non–Abelian black hole configuration in [18]. We can chose such a constant s for ${}^s\lambda$ when the effective source is zero (if ${}^s\lambda < 0$, this is possible for $\Lambda > 0$). The resulting nonholonomic matter field configurations $\mathbf{A}_\mu = {}^\circ\mathbf{A}_\mu + {}^\eta\mathbf{A}_\mu$ (37), $F_{\mu\nu} = s\sqrt{|\mathbf{g}|}\varepsilon_{\mu\nu}$ (38) and $\Phi = {}^\Phi\eta {}^\circ\Phi$ subjected to the conditions (39) are encoded as vacuum off–diagonal polarizations into solutions of equations (24)–(25).

5.1 (Non) holonomic non–Abelian effective vacuum spaces

This class of effective vacuum solutions are generated not just as a simple limit $\Lambda + {}^s\lambda \rightarrow 0$, for instance, for a class of solutions (42) with coefficients (43)–(45). We have to construct off–diagonal solutions of the Einstein equations for the canonical d–connection taking the vacuum equations $\widehat{\mathbf{R}}_{\alpha\beta} = 0$ and ansatz \mathbf{g} (36) with coefficients satisfying the conditions

$$\begin{aligned} \epsilon_1 \psi^{\bullet\bullet}(r, \theta) + \epsilon_2 \psi''(r, \theta) &= 0; \\ h_3 &= \pm e^{-2 {}^\circ\phi} \frac{(h_4^*)^2}{h_4} \text{ for a given } h_4(r, \theta, \varphi), \quad \phi(r, \theta, \varphi) = {}^\circ\phi = \text{const}; \\ w_i &= w_i(r, \theta, \varphi), \text{ for any such functions if } \lambda = 0; \\ n_i &= \begin{cases} {}^1n_i(r, \theta) + {}^2n_i(r, \theta) \int (h_4^*)^2 |h_4|^{-5/2} dv, & \text{if } n_i^* \neq 0; \\ {}^1n_i(r, \theta), & \text{if } n_i^* = 0. \end{cases} \end{aligned} \tag{72}$$

Effective vacuum solutions of the Einstein equations for the Levi–Civita connection, i.e of $R_{\alpha\beta} = 0$, are generated if we impose additional constraints

on coefficients of d-metric, for $e^{-2 \phi} = 1$, as solutions of (47),

$$h_3 = \pm 4 \left[\left(\sqrt{|h_4|} \right)^* \right]^2, \quad h_4^* \neq 0; \quad (73)$$

$$\begin{aligned} w_1 w_2 \left(\ln \left| \frac{w_1}{w_2} \right| \right)^* &= w_2^\bullet - w_1', \quad w_i^* \neq 0; \quad w_2^\bullet - w_1' = 0, \quad w_i^* = 0; \\ {}^1 n_1'(r, \theta) - {}^1 n_2^\bullet(r, \theta) &= 0, \quad n_i^* = 0. \end{aligned} \quad (74)$$

The constructed class of vacuum solutions with coefficients subjected to conditions (72)–(74) is of type (56) for (57)–(59). Such metrics consist a particular case of vacuum ansatz defined by Corollary 3.3 with $\underline{h}_4 = 1$ and $\omega = 1$.

Here we note that former analytic and numeric programs (for instance, standard ones with Maple/ Mathematica) for constructing solutions in gravity theories can not be directly applied for alternative verifications of our solutions. Those approaches do not encode the nonholonomic constraints which we use for constructing integral varieties. Nevertheless, it is possible to check in general analytic form, see all details summarized in Refs. [9, 10, 11] (and formulas from Appendix B), that the vacuum Einstein equations (24)–(15) with zero effective sources, ${}^v \Upsilon - 4s^2 = 0$ and $\Upsilon - 4s^2 = 0$, can be solved by above presented off-diagonal ansatz for metrics.

5.2 Non-Abelian deformations of the Schwarzschild metric

We can consider a "prime" metric which, in general, is not a solution of Einstein equations,

$${}^\varepsilon \mathbf{g} = -d\xi \otimes d\xi - r^2(\xi) d\vartheta \otimes d\vartheta - r^2(\xi) \sin^2 \vartheta d\varphi \otimes d\varphi + \varpi^2(\xi) dt \otimes dt. \quad (75)$$

We shall deform nonholonomically this metric into a "target" off-diagonal one which will be a solution of the vacuum Einstein equations. The non-trivial metric coefficients in (75) are stated in the form

$$\check{g}_1 = -1, \quad \check{g}_2 = -r^2(\xi), \quad \check{h}_3 = -r^2(\xi) \sin^2 \vartheta, \quad \check{h}_4 = \varpi^2(\xi), \quad (76)$$

for local coordinates $x^1 = \xi, x^2 = \vartheta, y^3 = \varphi, y^4 = t$, where

$$\xi = \int dr \left| 1 - \frac{2\mu_0}{r} + \frac{\varepsilon}{r^2} \right|^{1/2} \quad \text{and} \quad \varpi^2(r) = 1 - \frac{2\mu_0}{r} + \frac{\varepsilon}{r^2}.$$

If we put $\varepsilon = 0$ with μ_0 considered as a point mass, the metric ${}^\varepsilon \mathbf{g}$ (75) determines the Schwarzschild solution. For simplicity, we analyze only

the case of "pure" gravitational vacuum solutions, not considering a more general construction when $\varepsilon = e^2$ can be related to the electric charge for the Reissner–Nordström metric. In our approach, ε is a small parameter (eccentricity) defining a small deformation of a circle into an ellipse.

We generate exact solutions of the system (57)–(59) with effective $\Lambda + {}^s\lambda = 0$ via nonholonomic deformations ${}^\varepsilon\mathbf{g} \rightarrow {}^\varepsilon_\eta\mathbf{g}$, when $g_i = \eta_i \check{g}_i$ and $h_a = \eta_a \check{h}_a$ and w_i, n_i define a target metric

$$\begin{aligned} {}^\varepsilon_\eta\mathbf{g} &= \eta_1(\xi)d\xi \otimes d\xi + \eta_2(\xi)r^2(\xi) d\vartheta \otimes d\vartheta + \\ &\quad \eta_3(\xi, \vartheta, \varphi)r^2(\xi) \sin^2 \vartheta \delta\varphi \otimes \delta\varphi - \eta_4(\xi, \vartheta, \varphi)\varpi^2(\xi) \delta t \otimes \delta t, \\ \delta\varphi &= d\varphi + w_1(\xi, \vartheta, \varphi)d\xi + w_2(\xi, \vartheta, \varphi)d\vartheta, \quad \delta t = dt + n_1(\xi, \vartheta)d\xi + n_2(\xi, \vartheta)d\vartheta. \end{aligned} \quad (77)$$

The gravitational field equations for zero source relate the coefficients of the vertical metric and polarization functions,

$$h_3 = h_0^2(b^*)^2 = \eta_3(\xi, \vartheta, \varphi)r^2(\xi) \sin^2 \vartheta, \quad h_4 = -b^2 = -\eta_4(\xi, \vartheta, \varphi)\varpi^2(\xi), \quad (78)$$

for $|\eta_3| = (h_0)^2|\check{h}_4/\check{h}_3|[(\sqrt{|\eta_4|})^*]^2$. In these formulas, we have to chose $h_0 = \text{const}$ ($h_0 = 2$ in order to satisfy the first condition (74)), where \check{h}_a are stated by the Schwarzschild solution for the chosen system of coordinates and η_4 can be any function satisfying the condition $\eta_4^* \neq 0$. We generate a class of solutions for any function $b(\xi, \vartheta, \varphi)$ with $b^* \neq 0$. For different purposes, it is more convenient to work directly with η_4 , for $\eta_4^* \neq 0$, and/or h_4 , for $h_4^* \neq 0$. The gravitational polarizations η_1 and η_2 , when $\eta_1 = \eta_2 r^2 = e^{\psi(\xi, \vartheta)}$, are found from (24) with zero source, written in the form $\psi^{\bullet\bullet} + \psi'' = 0$.

Introducing the defined values of the coefficients in the ansatz (77), we find a class of exact off-diagonal vacuum solutions of the Einstein equations defining stationary nonholonomic deformations of the Schwarzschild metric,

$$\begin{aligned} {}^\varepsilon\mathbf{g} &= -e^\psi(d\xi \otimes d\xi + d\vartheta \otimes d\vartheta) - 4 \left[(\sqrt{|\eta_4|})^* \right]^2 \varpi^2 \delta\varphi \otimes \delta\varphi + \eta_4 \varpi^2 \delta t \otimes \delta t, \\ \delta\varphi &= d\varphi + w_1 d\xi + w_2 d\vartheta, \quad \delta t = dt + {}^1n_1 d\xi + {}^1n_2 d\vartheta. \end{aligned} \quad (79)$$

The N-connection coefficients $w_i(\xi, \vartheta, \varphi)$ and ${}^1n_i(\xi, \vartheta)$ must satisfy the conditions (74) in order to get vacuum metrics in GR.

It should be emphasized here that, in general, the bulk of solutions from the set of target metrics do not define black holes and do not describe obvious physical situations. They preserve the singular character of the coefficient ϖ^2 vanishing on the horizon of a Schwarzschild black hole if we take only smooth integration functions for some small deformation parameters ε .

5.3 Linear parametric polarizations induced by YMH fields

We may select some locally anisotropic configurations with possible physical interpretation of gravitational vacuum configurations with spherical and/or rotoid (ellipsoid) symmetry if it is considered a generating function

$$b^2 = q(\xi, \vartheta, \varphi) + \varepsilon \varrho(\xi, \vartheta, \varphi). \quad (80)$$

For simplicity, we restrict our analysis only with linear decompositions on a small parameter ε , with $0 < \varepsilon \ll 1$.

Using (80), we compute $(b^*)^2 = [(\sqrt{|q|})^*]^2 [1 + \varepsilon \frac{1}{(\sqrt{|q|})^*} (\varrho/\sqrt{|q|})^*]$ and the vertical coefficients of d-metric (79), i.e h_3 and h_4 (and corresponding polarizations η_3 and η_4), see formulas (78).¹¹ We model rotoid configurations if we chose

$$q = 1 - \frac{2\mu(\xi, \vartheta, \varphi)}{r} \text{ and } \varrho = \frac{q_0(r)}{4\mu^2} \sin(\omega_0 \varphi + \varphi_0), \quad (81)$$

for $\mu(\xi, \vartheta, \varphi) = \mu_0 + \varepsilon \mu_1(\xi, \vartheta, \varphi)$ (supposing that the mass is locally anisotropically polarized) with certain constants μ, ω_0 and φ_0 and arbitrary functions/polarizations $\mu_1(\xi, \vartheta, \varphi)$ and $q_0(r)$ to be determined from some boundary conditions, with ε being the eccentricity.¹² This condition defines a small deformation of the Schwarzschild spherical horizon into an ellipsoidal one (rotoid configuration with eccentricity ε).

The resulting off-diagonal solution with rotoid type symmetry is

$$\begin{aligned} {}^{rot}\mathbf{g} &= -e^\psi (d\xi \otimes d\xi + d\vartheta \otimes d\vartheta) + (q + \varepsilon \varrho) \delta t \otimes \delta t \\ &\quad - 4 \left[(\sqrt{|q|})^* \right]^2 \left[1 + \varepsilon \frac{1}{(\sqrt{|q|})^*} (\varrho/\sqrt{|q|})^* \right] \delta \varphi \otimes \delta \varphi, \quad (82) \\ \delta \varphi &= d\varphi + w_1 d\xi + w_2 d\vartheta, \quad \delta t = dt + {}^1n_1 d\xi + {}^1n_2 d\vartheta. \end{aligned}$$

The functions $q(\xi, \vartheta, \varphi)$ and $\varrho(\xi, \vartheta, \varphi)$ are given by formulas (81) and the N-connection coefficients $w_i(\xi, \vartheta, \varphi)$ and $n_i = {}^1n_i(\xi, \vartheta)$ are subjected to conditions of type (74),

$$\begin{aligned} w_1 w_2 \left(\ln \left| \frac{w_1}{w_2} \right| \right)^* &= w_2^\bullet - w_1', \quad w_i^* \neq 0; \\ \text{or } w_2^\bullet - w_1' &= 0, \quad w_i^* = 0; \quad {}^1n_1'(\xi, \vartheta) - {}^1n_2^\bullet(\xi, \vartheta) = 0 \end{aligned} \quad (83)$$

and $\psi(\xi, \vartheta)$ being any function for which $\psi^{\bullet\bullet} + \psi'' = 0$.

¹¹Nonholonomic deformations of the Schwarzschild solution (not depending on ε) can be generated if we consider $\varepsilon = 0$ and $b^2 = q$ and $(b^*)^2 = [(\sqrt{|q|})^*]^2$.

¹²We may treat ε as an eccentricity imposing the condition that the coefficient $h_4 = b^2 = \eta_4(\xi, \vartheta, \varphi) \varpi^2(\xi)$ becomes zero for data (81) if $r_+ \simeq 2\mu_0 / \left(1 + \varepsilon \frac{q_0(r)}{4\mu^2} \sin(\omega_0 \varphi + \varphi_0) \right)$.

6 Ellipsoidal EYMH Configurations and Solitons

We can consider nonholonomic deformations for the EYMH systems for arbitrary signs of the cosmological constant Λ and an effective nontrivial source $\Lambda + {}^s\lambda \neq 0$ containing contributions of nonholonomic YM configurations. Such classes of solutions can be constructed in general form for a system (24)–(27) and (47) with coefficients of metric of type (42). Such metrics consist a particular case of non-vacuum ansatz defined by Corollary 3.2 with $\underline{h}_4 = 1$ and $\omega = 1$.

6.1 Nonholonomic rotoid deformations

Using the anholonomic frame method, we can generate a class of solutions with nontrivial cosmological constant possessing different limits (for large radial distances and small nonholonomic deformations) than the vacuum configurations considered in previous section.

Let us consider a diagonal metric of type

$${}^\varepsilon_\lambda \mathbf{g} = d\xi \otimes d\xi + r^2(\xi) d\theta \otimes d\theta + r^2(\xi) \sin^2 \theta d\varphi \otimes d\varphi + {}_\lambda \varpi^2(\xi) dt \otimes dt, \quad (84)$$

where nontrivial metric coefficients are parametrized in the form $\check{g}_1 = 1$, $\check{g}_2 = r^2(\xi)$, $\check{h}_3 = r^2(\xi) \sin^2 \vartheta$, $\check{h}_4 = {}_\lambda \varpi^2(\xi)$, for local coordinates $x^1 = \xi, x^2 = \vartheta, y^3 = \varphi, y^4 = t$, with $\xi = \int dr / |q(r)|^{\frac{1}{2}}$, and ${}_\lambda \varpi^2(r) = -\sigma^2(r)q(r)$, for $q(r) = 1 - 2m(r)/r - \Lambda r^2/3$. In variables (r, θ, φ) , the metric (84) is equivalent to (B.7).

The ansatz for such classes of solutions is chosen in the form

$$\begin{aligned} {}^\lambda \check{\mathbf{g}} &= e^{\underline{\phi}(\xi, \theta)} (d\xi \otimes d\xi + d\theta \otimes d\theta) + h_3(\xi, \theta, \varphi) d\varphi \otimes d\varphi + h_4(\xi, \theta, \varphi) dt \otimes dt, \\ \delta\varphi &= d\varphi + w_1(\xi, \theta, \varphi) d\xi + w_2(\xi, \theta, \varphi) d\theta, \\ \delta t &= dt + n_1(\xi, \theta, \varphi) d\xi + n_2(\xi, \theta, \varphi) d\theta, \end{aligned}$$

for $h_3 = -h_0^2(b^*)^2 = \eta_3(\xi, \theta, \varphi)r^2(\xi) \sin^2 \vartheta$, $h_4 = b^2 = \eta_4(\xi, \theta, \varphi) {}_\lambda \varpi^2(\xi)$. The coefficients of this metric determine exact solutions if

$$\begin{aligned} \underline{\phi}^{\bullet\bullet}(\xi, \theta) + \underline{\phi}''(\xi, \theta) &= 2(\Lambda + {}^s\lambda); \quad (85) \\ h_3 &= \pm \frac{(\phi^*)^2}{4(\Lambda + {}^s\lambda)} e^{-2\phi(\xi, \theta)}, \quad h_4 = \mp \frac{1}{4(\Lambda + {}^s\lambda)} e^{2(\phi - \phi(\xi, \theta))}; \\ w_i &= -\partial_i \phi / \phi^*; \\ n_i &= {}^1n_i(\xi, \theta) + {}^2n_i(\xi, \theta) \int (\phi^*)^2 e^{-2(\phi - \phi(\xi, \theta))} d\varphi, \\ &= \begin{cases} {}^1n_i(\xi, \theta) + {}^2n_i(\xi, \theta) \int e^{-4\phi} \frac{(h_4^*)^2}{h_4} d\varphi, & \text{if } n_i^* \neq 0; \\ {}^1n_i(\xi, \theta), & \text{if } n_i^* = 0; \end{cases} \end{aligned}$$

for any nonzero h_a and h_a^* and (integrating) functions ${}^1n_i(\xi, \theta)$, ${}^2n_i(\xi, \theta)$, generating function $\phi(\xi, \theta, \varphi)$, and ${}^0\phi(\xi, \theta)$ to be determined from certain boundary conditions for a fixed system of coordinates.

For nonholonomic ellipsoid de Sitter configurations, we parametrize

$$\begin{aligned} {}^{rot}_\lambda \mathbf{g} &= -e^{\phi(\xi, \theta)} (d\xi \otimes d\xi + d\theta \otimes d\theta) + (\underline{q} + \varepsilon \underline{\varrho}) \delta t \otimes \delta t \\ &\quad - h_0^2 \left[(\sqrt{|\underline{q}|})^* \right]^2 \left[1 + \varepsilon \frac{1}{(\sqrt{|\underline{q}|})^*} (\underline{\varrho} / \sqrt{|\underline{q}|})^* \right] \delta \varphi \otimes \delta \varphi, \\ \delta \varphi &= d\varphi + w_1 d\xi + w_2 d\theta, \quad \delta t = dt + n_1 d\xi + n_2 d\theta, \end{aligned} \quad (86)$$

where $\underline{q} = 1 - \frac{2}{r} \frac{{}^1\mu(r, \theta, \varphi)}{\mu_0}$, $\underline{\varrho} = \frac{q_0(r)}{4\mu_0^2} \sin(\omega_0 \varphi + \varphi_0)$, are chosen to generate an anisotropic rotoid configuration for the smaller "horizon" (when $h_4 = 0$), $r_+ \simeq 2 \frac{{}^1\mu}{\mu_0} \left(1 + \varepsilon \frac{q_0(r)}{4\mu_0^2} \sin(\omega_0 \varphi + \varphi_0) \right)$, for a corresponding $q_0(r)$.

We have to impose the condition that the coefficients of the above d-metric induce a zero torsion in order to generate solutions of the Einstein equations for the Levi-Civita connection. Using formula (83), for $\phi^* \neq 0$, we obtain that $\phi(r, \varphi, \theta) = \ln |h_4^* / \sqrt{|h_3 h_4|}|$ must be any function defined in non-explicit form from equation $2e^{2\phi} \phi = \Lambda + {}^s\lambda$. The set of constraints for the N-connection coefficients is solved if the integration functions in (85) are chosen in a form when $w_1 w_2 \left(\ln \left| \frac{w_1}{w_2} \right| \right)^* = w_2^\bullet - w_1'$ for $w_i^* \neq 0$; $w_2^\bullet - w_1' = 0$ for $w_i^* = 0$; and take $n_i = {}^1n_i(x^k)$ for ${}^1n_1'(x^k) - {}^1n_2^\bullet(x^k) = 0$.

In a particular case, in the limit $\varepsilon \rightarrow 0$, we get a subclass of solutions of type (86) with spherical symmetry but with generic off-diagonal coefficients induced by the N-connection coefficients. This class of spacetimes depend on cosmological constants polarized nonholonomically by YMH fields. We can extract from such configurations the Schwarzschild solution if we select a set of functions with the properties $\phi \rightarrow const, w_i \rightarrow 0, n_i \rightarrow 0$ and $h_4 \rightarrow \varpi^2$. In general, the parametric dependence on cosmological constants and for effective YMH contributions, in nonholonomic configurations, is not smooth.

6.2 Effective vacuum solitonic configurations

It is possible to construct off-diagonal vacuum spacetimes generating by 3-d solitons as examples of generic off-diagonal solutions with nontrivial vertical conformal factor ω . We consider that there are satisfied the conditions of Corollary 3.3 with $\underline{h}_4 = 1$ for effective vacuum solutions (such configurations may encode EYMH fields) and the Cauchy problem is stated as in Section 3.

6.2.1 Solutions with solitonic factor $\omega(x^1, y^3, t)$

We take $\omega = \eta(x^1, y^3, t)$, when $y^4 = t$ is a time like coordinate, as a solution of KdP equation [24]

$$\pm \eta^{**} + (\partial_t \eta + \eta \eta^\bullet + \epsilon \eta^{\bullet\bullet\bullet})^\bullet = 0, \quad (87)$$

with dispersion ϵ and possible dependencies on a set of parameters θ . It is supposed that in the dispersionless limit $\epsilon \rightarrow 0$ the solutions are independent on y^3 and determined by Burgers' equation $\partial_t \eta + \eta \eta^\bullet = 0$. For such 3-d solitonic configurations, the conditions (47) are written in the form

$$\mathbf{e}_1 \eta = \eta^\bullet + w_1(x^i, y^3) \eta^* + n_1(x^i) \partial_t \eta = 0.$$

For $\eta' = 0$, we can impose the condition $w_2 = 0$ and $n_2 = 0$.

Such vacuum solitonic metrics can be parametrized in the form

$$\begin{aligned} \mathbf{g} &= e^{\psi(x^k)} (dx^1 \otimes dx^1 + dx^2 \otimes dx^2) + [\eta(x^1, y^3, t)]^2 h_a(x^1, y^3) \mathbf{e}^a \otimes \mathbf{e}^a, \\ \mathbf{e}^3 &= dy^3 + w_1(x^k, y^3) dx^1, \quad \mathbf{e}^4 = dy^4 + n_1(x^k) dx^1. \end{aligned}$$

This class of metrics does not have (in general) Killing symmetries but may possess symmetries determined by solitonic solutions of (87). Alternatively, we can consider that η is a solution of any three dimensional solitonic and/or other nonlinear wave equations; in a similar manner, we can generate solutions for $\omega = \eta(x^2, y^3, t)$.

6.2.2 Solitonic metrics with factor $\omega(x^i, t)$

There are effective vacuum metrics when the solitonic dynamics does not depend on anisotropic coordinate y^3 . In this case $\omega = \hat{\eta}(x^k, t)$ is a solution of KdP equation

$$\pm \hat{\eta}^{\bullet\bullet} + (\partial_t \hat{\eta} + \hat{\eta} \hat{\eta}' + \epsilon \hat{\eta}''')' = 0. \quad (88)$$

In the dispersionless limit $\epsilon \rightarrow 0$ the solutions are independent on x^1 and determined by Burgers' equation $\partial_t \hat{\eta} + \hat{\eta} \hat{\eta}' = 0$.

This class of vacuum solitonic EYM configurations is given by

$$\begin{aligned} {}_2\mathbf{g} &= e^{\psi(x^k)} (dx^1 \otimes dx^1 + dx^2 \otimes dx^2) + [\hat{\eta}(x^k, t)]^2 h_a(x^k, y^3) \mathbf{e}^a \otimes \mathbf{e}^a, \\ \mathbf{e}^3 &= dy^3 + w_1(x^k, y^3) dx^1, \quad \mathbf{e}^4 = dy^4 + n_1(x^k) dx^1; \end{aligned}$$

the conditions (47) are $\mathbf{e}_1 \hat{\eta} = \hat{\eta}^\bullet + n_1(x^i) \partial_t \hat{\eta} = 0$, $\mathbf{e}_2 \hat{\eta} = \hat{\eta}' + n_2(x^i) \partial_t \hat{\eta} = 0$.

It is possible to derive an infinite number of vacuum gravitational 2-d and 3-d configurations characterized by corresponding solitonic hierarchies and bi-Hamilton structures, for instance, related to different KdP equations (88) with possible mixtures with solutions for 2-d and 3-d sine-Gordon equations etc, see details in Ref. [25].

A Nonholonomic 2+2 Splitting of Lorentz Manifolds

In a general case, a metric-affine manifold V is endowed with a metric structure \mathbf{g} and an affine (linear) connection structure D (as a covariant differentiation operator). A linear connection gives us with the possibility to compute the directional derivative $D_X Y$ of a vector field Y in the direction of X . It is characterized by three fundamental geometric objects:

1. the torsion field is (by definition) $\mathcal{T}(X, Y) := D_{\mathbf{X}} Y - D_{\mathbf{Y}} X - [X, Y]$;
2. the curvature field is $\mathcal{R}(X, Y) := D_{\mathbf{X}} D_{\mathbf{Y}} - D_{\mathbf{Y}} D_{\mathbf{X}} - D_{[\mathbf{X}, \mathbf{Y}]}$;
3. the nonmetricity field is $\mathcal{Q}(X) := D_{\mathbf{X}} \mathbf{g}$.

Introducing $\mathbf{X} = \mathbf{e}_\alpha$ and $\mathbf{Y} = \mathbf{e}_\beta$, defined by (3), into above formulas, we compute the N-adapted coefficients and symmetries of $D = \{\Gamma^\gamma_{\alpha\beta}\}$ and corresponding fundamental geometric objects,

$$\begin{aligned}\mathcal{T} &= \{T^\gamma_{\alpha\beta} = (T^i_{jk}, T^i_{ja}, T^a_{ji}, T^a_{bi}, T^a_{bc})\}; \\ \mathcal{R} &= \{R^\alpha_{\beta\gamma\delta} = (R^i_{hjk}, R^a_{bjk}, R^i_{hja}, R^c_{bja}, R^i_{hba}, R^c_{bea})\}; \quad \mathcal{Q} = \{Q^\gamma_{\alpha\beta}\}.\end{aligned}$$

Every (pseudo) Riemannian manifold (V, \mathbf{g}) is naturally equipped with a Levi-Civita connection $D = \nabla = \{\nabla^\gamma \Gamma^\gamma_{\alpha\beta}\}$ completely defined by $\mathbf{g} = \{g_{\alpha\beta}\}$ if and only if there are satisfied the metric compatibility, $\nabla^\gamma \mathcal{Q}(X) = \nabla_{\mathbf{X}} \mathbf{g} = 0$, and zero torsion, $\nabla^\gamma \mathcal{T} = 0$, conditions. Hereafter, we shall write, for simplicity, $\nabla^\gamma \Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}$. It should be emphasized that ∇ does not preserve under parallelism and general frame/coordinate transforms a N-splitting (2). Nevertheless, it is possible to construct a unique distortion relation

$$\nabla = \hat{\mathbf{D}} + \hat{\mathbf{Z}}, \tag{A.1}$$

where both linear connections ∇ and $\hat{\mathbf{D}}$ (the second one can be considered as an auxiliary linear connection, which in literature is called the canonical distinguished connection; in brief, d-connection) and the distortion tensor

$\widehat{\mathbf{Z}}$, i.e. all values in the above formula, are completely defined by $\mathbf{g} = \{g_{\alpha\beta}\}$ for a prescribed $\mathbf{N} = \{N_i^a\}$, see details in [9, 10, 11].

Theorem A.1 *With respect to N -adapted frames (3) and (4), the coefficient of distortion relation (A.1) are computed*

$$\Gamma_{\alpha\beta}^\gamma = \widehat{\Gamma}_{\alpha\beta}^\gamma + \widehat{\mathbf{Z}}_{\alpha\beta}^\gamma, \quad (\text{A.2})$$

where the canonical d -connection $\widehat{\mathbf{D}} = \{ \widehat{\Gamma}_{\alpha\beta}^\gamma = (\widehat{L}_{jk}^i, \widehat{L}_{bk}^a, \widehat{C}_{jc}^i, \widehat{C}_{bc}^a) \}$ is defined by coefficients

$$\begin{aligned} \widehat{L}_{jk}^i &= \frac{1}{2}g^{ir}(\mathbf{e}_k g_{jr} + \mathbf{e}_j g_{kr} - \mathbf{e}_r g_{jk}), \\ \widehat{L}_{bk}^a &= e_b(N_k^a) + \frac{1}{2}g^{ac}(\mathbf{e}_k g_{bc} - g_{dc} e_b N_k^d - g_{db} e_c N_k^d), \\ \widehat{C}_{jc}^i &= \frac{1}{2}g^{ik}e_c g_{jk}, \quad \widehat{C}_{bc}^a = \frac{1}{2}g^{ad}(e_c g_{bd} + e_b g_{cd} - e_d g_{bc}), \end{aligned} \quad (\text{A.3})$$

and the distortion tensor $\widehat{\mathbf{Z}}_{\alpha\beta}^\gamma$ is

$$\begin{aligned} Z_{jk}^a &= -\widehat{C}_{jb}^i g_{ik} g^{ab} - \frac{1}{2}\Omega_{jk}^a, \quad Z_{bk}^i = \frac{1}{2}\Omega_{jk}^c g_{cb} g^{ji} - \Xi_{jk}^{ih} \widehat{C}_{hb}^j, \\ Z_{bk}^a &= +\Xi_{cd}^{ab} \widehat{T}_{kb}^c, \quad Z_{kb}^i = \frac{1}{2}\Omega_{jk}^a g_{cb} g^{ji} + \Xi_{jk}^{ih} \widehat{C}_{hb}^j, \quad Z_{jk}^i = 0, \\ Z_{jb}^a &= -\Xi_{cb}^{ad} \widehat{T}_{jd}^c, \quad Z_{bc}^a = 0, \quad Z_{ab}^i = -\frac{g^{ij}}{2} [\widehat{T}_{ja}^c g_{cb} + \widehat{T}_{jb}^c g_{ca}], \end{aligned} \quad (\text{A.4})$$

for $\Xi_{jk}^{ih} = \frac{1}{2}(\delta_j^i \delta_k^h - g_{jk} g^{ih})$ and $\pm \Xi_{cd}^{ab} = \frac{1}{2}(\delta_c^a \delta_d^b + g_{cd} g^{ab})$. The nontrivial coefficients Ω_{jk}^a and $\widehat{\mathbf{T}}_{\alpha\beta}^\gamma$ are given, respectively, by formulas (6) and, see below, (A.5).

Proof. It follows from a straightforward verification in N -adapted frames that the sums of coefficients (A.3) and (A.4) result in the coefficients of the Levi-Civita connection $\Gamma_{\alpha\beta}^\gamma$ for a general metric parametrized as a d -metric $\mathbf{g} = [g_{ij}, g_{ab}]$ (7). \square

All geometric constructions and physical theories derived for geometric data (\mathbf{g}, ∇) can be equivalently modeled by geometric data $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$ because of unique distortion relation (A.1).

Theorem A.2 *The nonholonomically induced torsion $\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}_{\alpha\beta}^\gamma\}$ of $\widehat{\mathbf{D}}$ is determined in a unique form by the metric compatibility condition, $\widehat{\mathbf{D}}\mathbf{g} = 0$,*

and zero horizontal and vertical torsion coefficients, $\hat{T}_{jk}^i = 0$ and $\hat{T}_{bc}^a = 0$, but with nontrivial h - v - coefficients

$$\hat{T}_{jk}^i = \hat{L}_{jk}^i - \hat{L}_{kj}^i, \hat{T}_{ja}^i = \hat{C}_{jb}^i, \hat{T}_{ji}^a = -\Omega_{ji}^a, \hat{T}_{aj}^c = \hat{L}_{aj}^c - e_a(N_j^c), \hat{T}_{bc}^a = \hat{C}_{bc}^a - \hat{C}_{cb}^a. \quad (\text{A.5})$$

Proof. The coefficients (A.5) are computed by introducing $D = \hat{\mathbf{D}}$, with coefficients (A.3), and $X = \mathbf{e}_\alpha, Y = \mathbf{e}_\beta$ (for N -adapted frames (3)) into standard formula for torsion, $\mathcal{T}(X, Y) := D_X Y - D_Y X - [X, Y]$.

□

In a similar form, introducing $\hat{\mathbf{D}}$ and $X = \mathbf{e}_\alpha, Y = \mathbf{e}_\beta, Z = \mathbf{e}_\gamma$ into $\mathcal{R}(X, Y) := D_X D_Y - D_Y D_X - D_{[X, Y]}$, we prove

Theorem A.3 *The curvature $\hat{\mathcal{R}} = \{\hat{\mathbf{R}}_{\beta\gamma\delta}^\alpha\}$ of $\hat{\mathbf{D}}$ is characterized by N -adapted coefficients*

$$\begin{aligned} \hat{R}_{hjk}^i &= e_k \hat{L}_{hj}^i - e_j \hat{L}_{hk}^i + \hat{L}_{hj}^m \hat{L}_{mk}^i - \hat{L}_{hk}^m \hat{L}_{mj}^i - \hat{C}_{ha}^i \Omega_{kj}^a, \\ \hat{R}_{bjk}^a &= e_k \hat{L}_{bj}^a - e_j \hat{L}_{bk}^a + \hat{L}_{bj}^c \hat{L}_{ck}^a - \hat{L}_{bk}^c \hat{L}_{cj}^a - \hat{C}_{bc}^a \Omega_{kj}^c, \\ \hat{R}_{jka}^i &= e_a \hat{L}_{jk}^i - \hat{D}_k \hat{C}_{ja}^i + \hat{C}_{jb}^i \hat{T}_{ka}^b, \hat{R}_{bka}^c = e_a \hat{L}_{bk}^c - D_k \hat{C}_{ba}^c + \hat{C}_{bd}^c \hat{T}_{ka}^d, \\ \hat{R}_{jbc}^i &= e_c \hat{C}_{jb}^i - e_b \hat{C}_{jc}^i + \hat{C}_{jb}^h \hat{C}_{hc}^i - \hat{C}_{jc}^h \hat{C}_{hb}^i, \\ \hat{R}_{bcd}^a &= e_d \hat{C}_{bc}^a - e_c \hat{C}_{bd}^a + \hat{C}_{bc}^e \hat{C}_{ed}^a - \hat{C}_{bd}^e \hat{C}_{ec}^a. \end{aligned} \quad (\text{A.6})$$

We can re-define the differential geometry of a (pseudo) Riemannian space \mathbf{V} in nonholonomic form in terms of geometric data $(\mathbf{g}, \hat{\mathbf{D}})$ which is equivalent to the "standard" formulation with (\mathbf{g}, ∇) .

Corollary A.1 *The Ricci tensor $\hat{\mathbf{R}}_{\alpha\beta} := \hat{\mathbf{R}}_{\alpha\beta\gamma}^\gamma$ (A.11) of $\hat{\mathbf{D}}$ is characterized by N -adapted coefficients*

$$\hat{\mathbf{R}}_{\alpha\beta} = \{\hat{R}_{ij} := \hat{R}_{ijk}^k, \hat{R}_{ia} := -\hat{R}_{ika}^k, \hat{R}_{ai} := \hat{R}_{aib}^b, \hat{R}_{ab} := \hat{R}_{abc}^c\}. \quad (\text{A.7})$$

Proof. The formulas for h - v -components (A.7) are obtained by contracting respectively the coefficients (A.6). Using $\hat{\mathbf{D}}$ (A.3), we express such formulas in terms of partial derivatives of coefficients of metric \mathbf{g} (1) and any equivalent parametrization in the form (7), or (8). □

The scalar curvature ${}^s\hat{R}$ of $\hat{\mathbf{D}}$ is by definition

$${}^s\hat{R} := \mathbf{g}^{\alpha\beta} \hat{\mathbf{R}}_{\alpha\beta} = g^{ij} \hat{R}_{ij} + g^{ab} \hat{R}_{ab}. \quad (\text{A.8})$$

Using (A.7) and (A.8), we can compute the Einstein tensor $\hat{\mathbf{E}}_{\alpha\beta}$ of $\hat{\mathbf{D}}$,

$$\hat{\mathbf{E}}_{\alpha\beta} \doteq \hat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2} \mathbf{g}_{\alpha\beta} {}^s\hat{R}. \quad (\text{A.9})$$

In general, this tensor is different from that constructed using (A.10) for the Levi–Civita connection ∇ .

Proposition A.1 *The N -adapted coefficients $\hat{\Gamma}_{\alpha\beta}^\gamma$ of $\hat{\mathbf{D}}$ are identic to the coefficients $\Gamma_{\alpha\beta}^\gamma$ of ∇ , both sets being computed with respect to N -adapted frames (3) and (4), if and only if there are satisfied the conditions $\hat{L}_{aj}^c = e_a(N_j^c)$, $\hat{C}_{jb}^i = 0$ and $\Omega_{ji}^a = 0$.*

Proof. If the conditions of the Proposition, i.e. constraints (A.13), are satisfied, all N -adapted coefficients of the torsion $\hat{\mathbf{T}}_{\alpha\beta}^\gamma$ (A.5) are zero. In such a case, the distortion tensor $\hat{\mathbf{Z}}_{\alpha\beta}^\gamma$ (A.4) is also zero. Following formula (A.2), we get $\Gamma_{\alpha\beta}^\gamma = \hat{\Gamma}_{\alpha\beta}^\gamma$. Inversely, if the last equalities of coefficients are satisfied for a chosen splitting (2), we get trivial torsions and distortions of ∇ . We emphasize that, in general, $\hat{\mathbf{D}} \neq \nabla$ because such connections have different transformation rules under frame/coordinate transforms. Nevertheless, if $\Gamma_{\alpha\beta}^\gamma = \hat{\Gamma}_{\alpha\beta}^\gamma$ in a N -adapted frame of reference, we get corresponding equalities for the Riemann and Ricci tensors etc. This means that the N -coefficients are such way fixed via frame transforms that the nonholonomic distribution became integrable even, in general, the frames (3) and (4) are nonholonomic (because not all anholonomy coefficients are not obligatory zero, for instance, $w_{ia}^b = \partial_a N_i^b$ may be nontrivial, see formulas (6)). \square

In order to elaborate models of gravity theories for ∇ and/or $\hat{\mathbf{D}}$, we have to consider the corresponding Ricci tensors,

$$Ric = \{R_{\beta\gamma} := R_{\beta\gamma\alpha}^\alpha\}, \text{ for } \nabla = \{\Gamma_{\alpha\beta}^\gamma\}, \quad (\text{A.10})$$

$$\text{and } \hat{Ric} = \{\hat{R}_{\beta\gamma} := \hat{R}_{\beta\gamma\alpha}^\alpha\}, \text{ for } \hat{\mathbf{D}} = \{\hat{\Gamma}_{\alpha\beta}^\gamma\}. \quad (\text{A.11})$$

Theorem A.4 *The field equations in GR can be re-written equivalently using the canonical d -connection $\hat{\mathbf{D}}$,*

$$\hat{\mathbf{R}}_{\beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta} {}^sR = \Upsilon_{\beta\delta}, \quad (\text{A.12})$$

$$\hat{L}_{aj}^c = e_a(N_j^c), \quad \hat{C}_{jb}^i = 0, \quad \Omega_{ji}^a = 0, \quad (\text{A.13})$$

where the scalar curvature ${}^sR := \mathbf{g}^{\beta\delta} \hat{\mathbf{R}}_{\beta\delta}$ and source d -tensor $\Upsilon_{\beta\delta}$ is such way constructed that $\Upsilon_{\beta\delta} \rightarrow \varkappa T_{\beta\delta}$ for $\hat{\mathbf{D}} \rightarrow \nabla$, where $T_{\beta\delta}$ is the energy-momentum tensor in GR with coupling gravitational constant \varkappa .

Proof. The Einstein equations are written in "standard" form for ∇ ,

$$E_{\beta\delta} = R_{\beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta} R = \kappa T_{\beta\delta}, \quad (\text{A.14})$$

where $R := \mathbf{g}^{\beta\delta} R_{\beta\delta}$. On spacetimes with conventional h- and v-splitting, we can define geometrically (or following a N-adapted variational calculus with operators (3) and (4)) a system of field equations with the Einstein tensor $\hat{\mathbf{E}}_{\alpha\beta}$ (A.9), written in the form (A.12). In general, both systems of PDE are different. But if the constraints (A.13) are imposed additionally on $\hat{\mathbf{D}}$, we satisfy the conditions of Proposition A.1, when $\Gamma_{\alpha\beta}^\gamma = \hat{\Gamma}_{\alpha\beta}^\gamma$ results in $R_{\beta\delta} = \hat{\mathbf{R}}_{\beta\delta}$ and $E_{\alpha\beta} = \hat{\mathbf{E}}_{\alpha\beta}$. The coefficients (A.5) are computed by introducing $D = \hat{\mathbf{D}}$, with coefficients (A.3), and $X = \mathbf{e}_\alpha, Y = \mathbf{e}_\beta$ (see (3)) into standard formula $\mathcal{T}(X, Y) := D_X Y - D_Y X - [X, Y]$. \square

We consider matter field sources in (A.12) which can be diagonalized with respect to N-adapted frames,

$$\Upsilon_\delta^\beta = \text{diag}[\Upsilon_\alpha : \Upsilon_1^1 = \Upsilon_2^2 = \Upsilon(x^k, y^3) + \underline{\Upsilon}(x^k, y^4); \Upsilon_3^3 = \Upsilon_4^4 = {}^v\Upsilon(x^k)]. \quad (\text{A.15})$$

Such a formal diagonalization can be performed via corresponding frame/coordinate transforms for very general distributions of matter fields.

An very important result which can be obtained using the anholonomic deformation method [9, 10, 11], and developed in this work, is that the Einstein equations for $\hat{\mathbf{D}}$ (A.12) decouple for parametrizations of d-metrics (7). The corresponding systems of PDE can be integrated in general forms with one Killing symmetry and for "non-Killing" configurations. This allows us to generate exact solutions of standard Einstein equations (A.14) in GR imposing additional nonholonomic constraints (A.13). It should be emphasized here that if we work from the very beginning with the Levi-Civita connection for metrics (8) computed with respect to coordinate, or other not N-adapted, frames, we can not "see" a possibility of general decoupling and "formal" integration of the gravitational field equations.

B Proof of Theorem 2.1

For $\omega = 1$ and $\underline{h}_a = \text{const}$, such proofs can be obtained by straightforward computations [9]. The approach was extended for $\omega \neq 1$ and higher dimensions in [10, 11]. In this section, we sketch a proof for ansatz (9) with nontrivial \underline{h}_4 depending on variable y^4 when $\omega = 1$ in data (10). At the next step, the formulas will be completed for nontrivial values $\omega \neq 1$.

If $\widehat{R}_1^1 = \widehat{R}_2^2$ and $\widehat{R}_3^3 = \widehat{R}_4^4$, the Einstein equations (A.12) for $\widehat{\mathbf{D}}$ and data (B.1) (see below) can be written for any source (A.15) in the form

$$\widehat{E}_1^1 = \widehat{E}_2^2 = -\widehat{R}_3^3 = \Upsilon(x^k, y^3) + \underline{\Upsilon}(x^k, y^3, y^4), \quad \widehat{E}_3^3 = \widehat{E}_4^4 = -\widehat{R}_1^1 = {}^v\Upsilon(x^k).$$

The geometric data for the conditions of Theorem 2.1 are $g_i = g_i(x^k)$ and

$$g_3 = h_3(x^k, y^3), g_4 = h_4(x^k, y^3)\underline{h}_4(x^k, y^4), N_i^3 = w_i(x^k, y^3), N_i^4 = n_i(x^k, y^3), \quad (\text{B.1})$$

for $\underline{h}_3 = 1$ and local coordinates $u^\alpha = (x^i, y^a) = (x^1, x^2, y^3, y^4)$. For such values, we shall compute respectively the coefficients of $\Omega_{\alpha\beta}^a$ in (6), canonical d-connection $\widehat{\Gamma}_{\alpha\beta}^\gamma$ (A.3), d-torsion $\widehat{\mathbf{T}}_{\alpha\beta}^\gamma$ (A.5), necessary coefficients of d-curvature $\widehat{\mathbf{R}}_{\alpha\beta\gamma}^\tau$ (A.6) with respective contractions for $\widehat{\mathbf{R}}_{\alpha\beta} := \widehat{\mathbf{R}}_{\alpha\beta\gamma}^\gamma$ (A.7) and resulting ${}^s\widehat{R}$ (A.8) and $\widehat{\mathbf{E}}_{\alpha\beta}$ (A.9). Finally, we shall state the conditions (A.13) when general coefficients (B.1) are considered for d-metrics.

B.1 Coefficients of the canonical d-connection

There are horizontal nontrivial coefficients of $\widehat{\Gamma}_{\alpha\beta}^\gamma$ (A.3),

$$\begin{aligned} \widehat{L}_{jk}^i &= \frac{1}{2}g^{i1}(\mathbf{e}_k g_{j1} + \mathbf{e}_j g_{k1} - \mathbf{e}_1 g_{jk}) + \frac{1}{2}g^{i2}(\mathbf{e}_k g_{j2} + \mathbf{e}_j g_{k2} - \mathbf{e}_2 g_{jk}) \\ &= \frac{1}{2}g^{i1}(\partial_k g_{j1} + \partial_j g_{k1} - \partial_1 g_{jk}) + \frac{1}{2}g^{i2}(\partial_k g_{j2} + \partial_j g_{k2} - \partial_2 g_{jk}), \\ \text{i.e. } \widehat{L}_{jk}^1 &= \frac{1}{2g_1}(\partial_k g_{j1} + \partial_j g_{k1} - g_{jk}^\bullet), \widehat{L}_{jk}^2 = \frac{1}{2g_2}(\partial_k g_{j2} + \partial_j g_{k2} - g_{jk}'). \end{aligned}$$

The h-v-components \widehat{L}_{bk}^a are computed following formulas

$$\begin{aligned} \widehat{L}_{bk}^3 &= e_b(N_k^3) + \frac{1}{2g_3}[\mathbf{e}_k g_{b3} - g_3 e_b N_k^3 - g_{3b}(N_k^3)^* - g_{4b}(N_k^4)^\circ] = e_b(N_k^3) \\ &\quad + \frac{1}{2g_3}[\partial_k g_{b3} - N_k^3 g_{b3}^* - N_k^4 g_{b3}^\circ - g_3 e_b N_k^3 - g_{3b}(N_k^3)^* - g_{4b}(N_k^4)^*] \\ &= e_b(w_k + \underline{w}_k) + \frac{1}{2g_3}[\partial_k g_{b3} - (w_k + \underline{w}_k)g_{b3}^* - (n_k + \underline{n}_k)g_{b3}^\circ \\ &\quad - g_3 e_b(w_k + \underline{w}_k) - g_{3b}w_k^* - g_{4b}n_k^*], \\ \widehat{L}_{bk}^4 &= e_b(N_k^4) + \frac{1}{2g_4}[\mathbf{e}_k g_{b4} - g_4 e_b N_k^4 - g_{3b}(N_k^3)^\circ - g_{4b}(N_k^4)^\circ] = e_b(N_k^4) \\ &\quad + \frac{1}{2g_4}[\partial_k g_{b4} - N_k^3 g_{b4}^* - N_k^4 g_{b4}^\circ - g_4 e_b N_k^4 - g_{3b}(N_k^3)^\circ - g_{4b}(N_k^4)^\circ] \\ &= e_b(n_k + \underline{n}_k) + \frac{1}{2g_4}[\partial_k g_{b4} - (w_k + \underline{w}_k)g_{b4}^* - (n_k + \underline{n}_k)g_{b4}^\circ \\ &\quad - g_4 e_b(n_k + \underline{n}_k) - g_{3b}\underline{w}_k^\circ - g_{4b}\underline{n}_k^\circ]. \end{aligned}$$

In explicit form, we obtain these nontrivial values

$$\begin{aligned}
\hat{L}_{3k}^3 &= w_k^* + \frac{1}{2g_3} [\partial_k g_3 - w_k] g_3^* - n_k g_3^\circ - g_3 w_k^* - g_3 w_k^*] \\
&= \frac{1}{2g_3} [\partial_k g_3 - w_k g_3^*] = \frac{\partial_k h_3}{2h_3} - w_k \frac{h_3^*}{2h_3}, \\
\hat{L}_{4k}^3 &= \frac{1}{2g_3} [-g_4 n_k^*] = -\frac{h_4 \underline{h}_4}{2h_3} n_k^*, \quad \hat{L}_{3k}^4 = n_k^* + \frac{1}{2g_4} [-g_4 n_k^*] = \frac{1}{2} n_k^*, \\
\hat{L}_{4k}^4 &= \frac{1}{2g_4} [\partial_k g_4 - w_k g_4^* - n_k g_4^\circ] = \frac{\partial_k (h_4 \underline{h}_4)}{2h_4 \underline{h}_4} - w_k \frac{h_4^*}{2h_4} - n_k \frac{\underline{h}_4^\circ}{2\underline{h}_4}.
\end{aligned}$$

For the set of h - v C -coefficients, we get $\hat{C}_{jc}^i = \frac{1}{2} g^{ik} \frac{\partial g_{jk}}{\partial y^c} = 0$. The v -components of C -coefficients are computed following formulas

$$\hat{C}_{bc}^3 = \frac{1}{2g_3} (e_c g_{b3} + e_c g_{c3} - g_{bc}^*), \quad \hat{C}_{bc}^4 = \frac{1}{2g_4} (e_c g_{b4} + e_b g_{c4} - g_{bc}^\circ),$$

i. e. $\hat{C}_{33}^3 = \frac{g_3^*}{2g_3} = \frac{h_3^*}{2h_3}$, $\hat{C}_{34}^3 = \frac{g_3^\circ}{2g_3} = 0$, $\hat{C}_{44}^3 = -\frac{g_4^*}{2g_3} = -\frac{h_4^* \underline{h}_4}{h_3}$, $\hat{C}_{33}^4 = -\frac{g_3^\circ}{2g_4} = 0$, $\hat{C}_{34}^4 = \frac{g_4^*}{2g_4} = \frac{h_4^*}{2h_4}$, $\hat{C}_{44}^4 = \frac{g_4^\circ}{2g_4} = \frac{\underline{h}_4^\circ}{2\underline{h}_4}$. Putting together the above formulas, we find all nontrivial coefficients,

$$\begin{aligned}
\hat{L}_{11}^1 &= \frac{g_1^\bullet}{2g_1}, \quad \hat{L}_{12}^1 = \frac{g_1'}{2g_1}, \quad \hat{L}_{22}^1 = -\frac{g_2^\bullet}{2g_1}, \quad \hat{L}_{11}^2 = \frac{-g_1'}{2g_2}, \quad \hat{L}_{12}^2 = \frac{g_2^\bullet}{2g_2}, \quad \hat{L}_{22}^2 = \frac{g_2'}{2g_2}, \quad (\text{B.2}) \\
\hat{L}_{4k}^4 &= \frac{\partial_k (h_4 \underline{h}_4)}{2h_4 \underline{h}_4} - \frac{w_k h_4^*}{2h_4} - (n_k + \underline{n}_k) \frac{\underline{h}_4^\circ}{2\underline{h}_4}, \quad \hat{L}_{3k}^3 = \frac{\partial_k h_3}{2h_3} - \frac{w_k h_3^*}{2h_3}, \quad \hat{L}_{4k}^3 = \frac{h_4 \underline{h}_4}{-2h_3} n_k^*, \\
\hat{L}_{3k}^4 &= \frac{1}{2} n_k^*, \quad \hat{C}_{33}^3 = \frac{h_3^*}{2h_3}, \quad \hat{C}_{44}^3 = -\frac{h_4^* \underline{h}_4}{h_3}, \quad \hat{C}_{33}^4 = -\frac{h_3 \underline{h}_3^\circ}{h_4 \underline{h}_4}, \quad \hat{C}_{34}^4 = \frac{h_4^*}{2h_4}, \quad \hat{C}_{44}^4 = \frac{\underline{h}_4^\circ}{2\underline{h}_4}.
\end{aligned}$$

We shall need also the values

$$\hat{C}_3 = \hat{C}_{33}^3 + \hat{C}_{34}^4 = \frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4}, \quad \hat{C}_4 = \hat{C}_{43}^3 + \hat{C}_{44}^4 = \frac{\underline{h}_4^\circ}{2\underline{h}_4}. \quad (\text{B.3})$$

B.2 Coefficients for torsion of $\hat{\mathbf{D}}$

Using data (B.1) for $\underline{w}_i = \underline{n}_i = 0$, the coefficients $\Omega_{ij}^a = \mathbf{e}_j (N_i^a) - \mathbf{e}_i (N_j^a)$ (6), are computed

$$\begin{aligned}
\Omega_{ij}^a &= \partial_j (N_i^a) - \partial_i (N_j^a) - N_i^b \partial_b N_j^a + N_j^b \partial_b N_i^a \\
&= \partial_j (N_i^a) - \partial_i (N_j^a) - N_i^3 (N_j^a)^* - N_i^4 (N_j^a)^\circ + N_j^3 (N_i^a)^* + N_j^4 (N_i^a)^\circ \\
&= \partial_j (N_i^a) - \partial_i (N_j^a) - w_i (N_j^a)^* + w_j (N_i^a)^*.
\end{aligned}$$

We obtain such nontrivial values

$$\begin{aligned}\Omega_{12}^3 &= -\Omega_{21}^3 = \partial_2 w_1 - \partial_1 w_2 - w_1 w_2^* + w_2 w_1^* = w_1' - w_2^\bullet - w_1 w_2^* + w_2 w_1^*; \\ \Omega_{12}^4 &= -\Omega_{21}^4 = \partial_2 n_1 - \partial_1 n_2 - w_1 n_2^* + w_2 n_1^* = n_1' - n_2^\bullet - w_1 n_2^* + w_2 n_1^*. \quad (\text{B.4})\end{aligned}$$

The nontrivial coefficients of d-torsion (A.5) are $\hat{T}_{ji}^a = -\Omega_{ji}^a$ (B.4) and $\hat{T}_{aj}^c = \hat{L}_{aj}^c - e_a(N_j^c)$. We find zero values for other types of coefficients,

$$\hat{T}_{jk}^i = \hat{L}_{jk}^i - \hat{L}_{kj}^i = 0, \quad \hat{T}_{ja}^i = \hat{C}_{jb}^i = 0, \quad \hat{T}_{bc}^a = \hat{C}_{bc}^a - \hat{C}_{cb}^a = 0.$$

We have such nontrivial N-adapted coefficients of d-torsion:

$$\begin{aligned}\hat{T}_{3k}^3 &= \hat{L}_{3k}^3 - e_3(N_k^3) = \frac{\partial_k h_3}{2h_3} - w_k \frac{h_3^*}{2h_3} - w_k^*, \\ \hat{T}_{4k}^3 &= \hat{L}_{4k}^3 - e_4(N_k^3) = -\frac{h_4 \underline{h}_4}{2h_3} n_k^*, \quad \hat{T}_{3k}^4 = \hat{L}_{3k}^4 - e_3(N_k^4) = \frac{1}{2} n_k^* - n_k^* = -\frac{1}{2} n_k^*, \\ \hat{T}_{4k}^4 &= \hat{L}_{4k}^4 - e_4(N_k^4) = \frac{\partial_k (h_4 \underline{h}_4)}{2h_4 \underline{h}_4} - w_k \frac{h_4^*}{2h_4} - n_k \frac{\underline{h}_4^\circ}{2\underline{h}_4}, \\ -\hat{T}_{12}^3 &= w_1' - w_2^\bullet - w_1 w_2^* + w_2 w_1^*, \quad -\hat{T}_{12}^4 = n_1' - n_2^\bullet - w_1 n_2^* + w_2 n_1^*. \quad (\text{B.5})\end{aligned}$$

If all coefficients (B.5) are zero, $\Gamma_{\alpha\beta}^\gamma = \hat{\mathbf{\Gamma}}_{\alpha\beta}^\gamma$.

B.3 Calculation of the Ricci tensor

Let us compute the values $\hat{R}_{ij} = \hat{R}_{ijk}^k$ from (A.11) using (A.6),

$$\begin{aligned}\hat{R}_{hjk}^i &= \mathbf{e}_k \hat{L}_{hj}^i - \mathbf{e}_j \hat{L}_{hk}^i + \hat{L}_{hj}^m \hat{L}_{mk}^i - \hat{L}_{hk}^m \hat{L}_{mj}^i - \hat{C}_{ha}^i \Omega_{jk}^a \\ &= \partial_k \hat{L}_{hj}^i - \partial_j \hat{L}_{hk}^i + \hat{L}_{hj}^m \hat{L}_{mk}^i - \hat{L}_{hk}^m \hat{L}_{mj}^i,\end{aligned}$$

where we put $\hat{C}_{ha}^i = 0$ and

$$\mathbf{e}_k \hat{L}_{hj}^i = \partial_k \hat{L}_{hj}^i + N_k^a \partial_a \hat{L}_{hj}^i = \partial_k \hat{L}_{hj}^i + w_k \left(\hat{L}_{hj}^i \right)^* + n_k \left(\hat{L}_{hj}^i \right)^\circ = \partial_k \hat{L}_{hj}^i.$$

Taking derivatives of (B.2), we obtain

$$\begin{aligned}\partial_1 \hat{L}_{11}^1 &= \left(\frac{g_1^\bullet}{2g_1} \right)^\bullet = \frac{g_1^{\bullet\bullet}}{2g_1} - \frac{(g_1^\bullet)^2}{2(g_1)^2}, \quad \partial_1 \hat{L}_{12}^1 = \left(\frac{g_1'}{2g_1} \right)^\bullet = \frac{g_1'^\bullet}{2g_1} - \frac{g_1^\bullet g_1'}{2(g_1)^2}, \\ \partial_1 \hat{L}_{22}^1 &= \left(-\frac{g_2^\bullet}{2g_1} \right)^\bullet = -\frac{g_2^{\bullet\bullet}}{2g_1} + \frac{g_1^\bullet g_2^\bullet}{2(g_1)^2}, \quad \partial_1 \hat{L}_{11}^2 = \left(-\frac{g_1'}{2g_2} \right)^\bullet = -\frac{g_1'^\bullet}{2g_2} + \frac{g_1^\bullet g_2'}{2(g_2)^2}, \\ \partial_1 \hat{L}_{12}^2 &= \left(\frac{g_2^\bullet}{2g_2} \right)^\bullet = \frac{g_2^{\bullet\bullet}}{2g_2} - \frac{(g_2^\bullet)^2}{2(g_2)^2}, \quad \partial_1 \hat{L}_{22}^2 = \left(\frac{g_2'}{2g_2} \right)^\bullet = \frac{g_2'^\bullet}{2g_2} - \frac{g_2^\bullet g_2'}{2(g_2)^2},\end{aligned}$$

$$\begin{aligned}
\partial_2 \hat{L}^1_{11} &= \left(\frac{g_1^\bullet}{2g_1}\right)' = \frac{g_1^{\bullet'}}{2g_1} - \frac{g_1^\bullet g_1'}{2(g_1)^2}, \quad \partial_2 \hat{L}^1_{12} = \left(\frac{g_1'}{2g_1}\right)' = \frac{g_1''}{2g_1} - \frac{(g_1')^2}{2(g_1)^2}, \\
\partial_2 \hat{L}^1_{22} &= \left(-\frac{g_2^\bullet}{2g_1}\right)' = -\frac{g_2^{\bullet'}}{2g_1} + \frac{g_2^\bullet g_1'}{2(g_1)^2}, \quad \partial_2 \hat{L}^2_{11} = \left(-\frac{g_1'}{2g_2}\right)' = -\frac{g_1''}{2g_2} + \frac{g_1^\bullet g_1'}{2(g_2)^2}, \\
\partial_2 \hat{L}^2_{12} &= \left(\frac{g_2^\bullet}{2g_2}\right)' = \frac{g_2^{\bullet'}}{2g_2} - \frac{g_2^\bullet g_2'}{2(g_2)^2}, \quad \partial_2 \hat{L}^2_{22} = \left(\frac{g_2'}{2g_2}\right)' = \frac{g_2''}{2g_2} - \frac{(g_2')^2}{2(g_2)^2}.
\end{aligned}$$

For these values, there are only 2 nontrivial components,

$$\begin{aligned}
\hat{R}^1_{212} &= \frac{g_2^{\bullet\bullet}}{2g_1} - \frac{g_1^\bullet g_2^\bullet}{4(g_1)^2} - \frac{(g_2^\bullet)^2}{4g_1 g_2} + \frac{g_1''}{2g_1} - \frac{g_1' g_2'}{4g_1 g_2} - \frac{(g_1')^2}{4(g_1)^2}, \\
\hat{R}^2_{112} &= -\frac{g_2^{\bullet\bullet}}{2g_2} + \frac{g_1^\bullet g_2^\bullet}{4g_1 g_2} + \frac{(g_2^\bullet)^2}{4(g_2)^2} - \frac{g_1''}{2g_2} + \frac{g_1' g_2'}{4(g_2)^2} + \frac{(g_1')^2}{4g_1 g_2}.
\end{aligned}$$

Considering $\hat{R}_{11} = -\hat{R}^2_{112}$ and $\hat{R}_{22} = \hat{R}^1_{212}$, when $g^i = 1/g_i$, we find

$$\hat{R}^1_{11} = \hat{R}^2_{22} = -\frac{1}{2g_1 g_2} [g_2^{\bullet\bullet} - \frac{g_1^\bullet g_2^\bullet}{2g_1} - \frac{(g_2^\bullet)^2}{2g_2} + g_1'' - \frac{g_1' g_2'}{2g_2} - \frac{(g_1')^2}{2g_1}],$$

which can be found in equations (25).

The next step is to derive the equations (26). We consider the third formula in (A.6),

$$\begin{aligned}
\hat{R}^c_{bka} &= \frac{\partial \hat{L}^c_{bk}}{\partial y^a} - \left(\frac{\partial \hat{C}^c_{ba}}{\partial x^k} + \hat{L}^c_{dk} \hat{C}^d_{ba} - \hat{L}^d_{bk} \hat{C}^c_{da} - \hat{L}^d_{ak} \hat{C}^c_{bd}\right) + \hat{C}^c_{bd} \hat{T}^d_{ka} \\
&= \frac{\partial \hat{L}^c_{bk}}{\partial y^a} - \hat{C}^c_{ba|k} + \hat{C}^c_{bd} \hat{T}^d_{ka}.
\end{aligned}$$

Contracting indices, we get $\hat{R}_{bk} = \hat{R}^a_{bka} = \frac{\partial L^a_{bk}}{\partial y^a} - \hat{C}^a_{ba|k} + \hat{C}^a_{bd} \hat{T}^d_{ka}$. For $\hat{C}_b := \hat{C}^c_{ba}$, we write

$$\begin{aligned}
\hat{C}_{b|k} &= \mathbf{e}_k \hat{C}_b - \hat{L}^d_{bk} \hat{C}_d = \partial_k \hat{C}_b - N^e_k \partial_e \hat{C}_b - \hat{L}^d_{bk} \hat{C}_d \\
&= \partial_k \hat{C}_b - w_k \hat{C}_b^* - n_k \hat{C}_b^\circ - \hat{L}^d_{bk} \hat{C}_d.
\end{aligned}$$

We split conventionally $\hat{R}_{bk} = {}_{[1]}R_{bk} + {}_{[2]}R_{bk} + {}_{[3]}R_{bk}$, where

$$\begin{aligned}
{}_{[1]}R_{bk} &= \left(\hat{L}^3_{bk}\right)^* + \left(\hat{L}^4_{bk}\right)^\circ, \quad {}_{[2]}R_{bk} = -\partial_k \hat{C}_b + w_k \hat{C}_b^* + n_k \hat{C}_b^\circ + \hat{L}^d_{bk} \hat{C}_d, \\
{}_{[3]}R_{bk} &= \hat{C}^a_{bd} \hat{T}^d_{ka} = \hat{C}^3_{b3} \hat{T}^3_{k3} + \hat{C}^3_{b4} \hat{T}^4_{k3} + \hat{C}^4_{b3} \hat{T}^3_{k4} + \hat{C}^4_{b4} \hat{T}^4_{k4}.
\end{aligned}$$

Using formulas (B.2), (B.5) and (B.3), we compute

$$\begin{aligned}
{}_{[1]}R_{3k} &= \left(\widehat{L}_{3k}^3\right)^* + \left(\widehat{L}_{3k}^4\right)^\circ = \left(\frac{\partial_k h_3}{2h_3} - w_k \frac{h_3^*}{2h_3}\right)^* \\
&= -w_k^* \frac{h_3^*}{2h_3} - w_k \left(\frac{h_3^*}{2h_3}\right)^* + \frac{1}{2} \left(\frac{\partial_k h_3}{h_3}\right)^*, \\
{}_{[2]}R_{3k} &= -\partial_k \widehat{C}_3 + w_k \widehat{C}_3^* + n_k \widehat{C}_3^\circ + \widehat{L}_{3k}^3 \widehat{C}_3 + \widehat{L}_{3k}^4 \widehat{C}_4 = -\partial_k \left(\frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4}\right) + \\
&\quad w_k \left(\frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4}\right)^* + \left(\frac{\partial_k h_3}{2h_3} - w_k \frac{h_3^*}{2h_3}\right) \left(\frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4}\right) + \frac{1}{2} n_k^* \frac{h_4^\circ}{2h_4} \\
&= w_k \left[\frac{h_3^*}{2h_3} - \frac{3}{4} \frac{(h_3^*)^2}{(h_3)^2} + \frac{h_4^*}{2h_4} - \frac{1}{2} \frac{(h_4^*)^2}{(h_4)^2} - \frac{1}{4} \frac{h_3^* h_4^*}{h_3 h_4}\right] + n_k^* \frac{h_4^\circ}{4h_4} \\
&\quad + \left(\frac{\partial_k h_3}{2h_3} + \frac{\partial_k h_4}{2h_4}\right) \left(\frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4}\right) - \frac{1}{2} \partial_k \left(\frac{h_3^*}{h_3} + \frac{h_4^*}{h_4}\right), \\
{}_{[3]}R_{3k} &= \widehat{C}_{33}^3 \widehat{T}_{k3}^3 + \widehat{C}_{34}^3 \widehat{T}_{k3}^4 + \widehat{C}_{33}^4 \widehat{T}_{k4}^3 + \widehat{C}_{34}^4 \widehat{T}_{k4}^4 \\
&= -\frac{h_3^*}{2h_3} \left[\frac{\partial_k h_3}{2h_3} - w_k \frac{h_3^*}{2h_3} - w_k^*\right] - \frac{h_4^*}{2h_4} \left[\frac{\partial_k (h_4 h_4)}{2h_4 h_4} - w_k \frac{h_4^*}{2h_4} - n_k \frac{h_4^\circ}{2h_4}\right] \\
&= w_k^* \frac{h_3^*}{2h_3} + w_k \left(\frac{(h_3^*)^2}{4(h_3)^2} + \frac{(h_4^*)^2}{4(h_4)^2}\right) n_k \frac{h_4^*}{2h_4} \frac{h_4^\circ}{2h_4} \\
&\quad - \frac{h_3^*}{2h_3} \frac{\partial_k h_3}{2h_3} - \frac{h_4^*}{2h_4} \left(\frac{\partial_k h_4}{2h_4} + \frac{\partial_k h_4}{2h_4}\right).
\end{aligned}$$

Summarizing, we get

$$\begin{aligned}
\widehat{R}_{3k} &= w_k \left[\frac{h_4^*}{2h_4} - \frac{1}{4} \frac{(h_4^*)^2}{(h_4)^2} - \frac{1}{4} \frac{h_3^* h_4^*}{h_3 h_4}\right] + n_k^* \frac{h_4^\circ}{4h_4} + n_k \frac{h_4^*}{2h_4} \frac{h_4^\circ}{2h_4} \\
&\quad + \frac{h_4^*}{2h_4} \frac{\partial_k h_3}{2h_3} - \frac{1}{2} \frac{\partial_k h_4^*}{h_4} + \frac{1}{4} \frac{h_4^* \partial_k h_4}{(h_4)^2} - \frac{h_4^*}{2h_4} \frac{\partial_k h_4}{2h_4}.
\end{aligned}$$

which is equivalent to (26) if the conditions $n_k h_4^\circ = \partial_k h_4$, see below formula (B.6), are satisfied.

In a similar way, we compute $\widehat{R}_{4k} = {}_{[1]}R_{4k} + {}_{[2]}R_{4k} + {}_{[3]}R_{4k}$, where

$$\begin{aligned}
{}_{[1]}R_{4k} &= \left(\widehat{L}_{4k}^3\right)^* + \left(\widehat{L}_{4k}^4\right)^\circ, \quad {}_{[2]}R_{4k} = -\partial_k \widehat{C}_4 + w_k \widehat{C}_4^* + n_k \widehat{C}_4^\circ + \widehat{L}_{4k}^3 \widehat{C}_3 \\
&\quad + \widehat{L}_{4k}^4 \widehat{C}_4, \quad {}_{[3]}R_{4k} = \widehat{C}_{4d}^a \widehat{T}_{ka}^d = \widehat{C}_{43}^3 \widehat{T}_{k3}^3 + \widehat{C}_{44}^3 \widehat{T}_{k3}^4 + \widehat{C}_{43}^4 \widehat{T}_{k4}^3 + \widehat{C}_{44}^4 \widehat{T}_{k4}^4.
\end{aligned}$$

For the first term, we use respective coefficients of d-connection, \widehat{L}_{4k}^3 and

\widehat{L}_{4k}^4 , from (B.2),

$$\begin{aligned} [1]R_{4k} &= \left(\widehat{L}_{4k}^3\right)^* + \left(\widehat{L}_{4k}^4\right)^\circ = -\left(\frac{h_4\hbar_4}{2h_3}n_k^*\right)^* + \left(\frac{\partial_k(h_4\hbar_4)}{2h_4\hbar_4} - w_k\frac{h_4^*}{2h_4} - n_k\right)\frac{\hbar_4^\circ}{2\hbar_4}^\circ \\ &= -n_k^{**}\frac{h_4}{2h_3}\hbar_4 - n_k^*\left(\frac{h_4^*}{2h_3} - \frac{h_4^*h_3^*}{2(h_3)^*}\right)\hbar_4 - n_k\left(\frac{\hbar_4^\circ}{2\hbar_4} - \frac{(\hbar_4^\circ)^2}{2(\hbar_4)^2}\right) + \frac{\partial_k\hbar_4^\circ}{2\hbar_4} - \frac{\hbar_4^\circ\partial_k\hbar_4}{2(\hbar_4)^2}. \end{aligned}$$

In order to compute the second term, we use formulas (B.3), for \widehat{C}_3 and \widehat{C}_4 , and (B.2), for \widehat{L}_{4k}^3 and \widehat{L}_{4k}^4 ,

$$\begin{aligned} [2]R_{4k} &= -\partial_k\widehat{C}_4 + w_k\widehat{C}_4^* + n_k\widehat{C}_4^\circ + \widehat{L}_{4k}^3\widehat{C}_3 + \widehat{L}_{4k}^4\widehat{C}_4 \\ &= -\partial_k\left(\frac{\hbar_4^\circ}{2\hbar_4}\right) + n_k\left(\frac{\hbar_4^\circ}{2\hbar_4}\right)^\circ + -\frac{h_4\hbar_4}{2h_3}n_k^*\left(\frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4}\right) \\ &\quad + \left(\frac{\partial_k(h_4\hbar_4)}{2h_4\hbar_4} - w_k\frac{h_4^*}{2h_4} - n_k\frac{\hbar_4^\circ}{2\hbar_4}\right)\frac{\hbar_4^\circ}{2\hbar_4} \\ &= -w_k\left(\frac{h_4^*}{2h_4}\frac{\hbar_4^\circ}{2\hbar_4}\right) - n_k^*\frac{h_4\hbar_4}{2h_3\hbar_3}\left(\frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4}\right) \\ &\quad + n_k\left[\left(\frac{\hbar_4^\circ}{2\hbar_4}\right)^\circ - \frac{\hbar_4^\circ}{2\hbar_4}\frac{\hbar_4^\circ}{2\hbar_4}\right] + \frac{\partial_k h_4}{2h_4}\frac{\hbar_4^\circ}{2\hbar_4} + \frac{\partial_k\hbar_4}{2\hbar_4}\frac{\hbar_4^\circ}{2\hbar_4} - \frac{\partial_k\hbar_4^\circ}{2\hbar_4} + \frac{\hbar_4^\circ\partial_k\hbar_4}{2(\hbar_4)^2}. \end{aligned}$$

For the third term we use formulas (B.2), with $\widehat{C}_{43}^3, \widehat{C}_{44}^3, \widehat{C}_{43}^4, \widehat{C}_{44}^4$, and the formulas (B.5), with $\widehat{T}_{k3}^3, \widehat{T}_{k3}^4, \widehat{T}_{k4}^3, \widehat{T}_{k4}^4$,

$$\begin{aligned} [3]R_{4k} &= \widehat{C}_{43}^3\widehat{T}_{k3}^3 + \widehat{C}_{44}^3\widehat{T}_{k3}^4 + \widehat{C}_{43}^4\widehat{T}_{k4}^3 + \widehat{C}_{44}^4\widehat{T}_{k4}^4 \\ &= -\frac{\hbar_3^\circ}{2\hbar_3}(-w_k^* - w_k\frac{h_3^*}{2h_3} + \frac{\partial_k h_3}{2h_3}) + \frac{h_4^*\hbar_4}{2h_3}(-\frac{1}{2}n_k^*) - \frac{h_4^*}{2h_4}(-\frac{h_4\hbar_4}{2h_3}n_k^*) \\ &\quad - \frac{\hbar_4^\circ}{2\hbar_4}\left(-w_k\frac{h_4^*}{2h_4} - n_k\frac{\hbar_4^\circ}{2\hbar_4} + \frac{\partial_k h_4}{2h_4} + \frac{\partial_k\hbar_4}{2\hbar_4}\right) \\ &= w_k\frac{h_4^*}{2h_4}\frac{\hbar_4^\circ}{2\hbar_4} + n_k\left(\frac{\hbar_4^\circ}{2\hbar_4}\right)^2 - \frac{\partial_k h_4}{2h_4}\frac{\hbar_4^\circ}{2\hbar_4} - \frac{\hbar_4^\circ}{2\hbar_4}\frac{\partial_k\hbar_4}{2\hbar_4}. \end{aligned}$$

Taking the last equalities in above three formulas, we get

$$\begin{aligned} [1]R_{4k} &= -n_k^{**}\frac{h_4}{2h_3}\hbar_4 + n_k^*\left(-\frac{h_4^*}{2h_3} + \frac{h_4^*h_3^*}{2(h_3)^2}\right)\hbar_4 + \frac{\partial_k\hbar_4^\circ}{2\hbar_4} - \frac{\hbar_4^\circ\partial_k\hbar_4}{2(\hbar_4)^2}, \\ [2]R_{4k} &= w_k\left(-\frac{h_4^*}{2h_4}\frac{\hbar_4^\circ}{2\hbar_4}\right) - n_k^*\frac{h_4\hbar_4}{2h_3\hbar_3}\left(\frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4}\right) + n_k\left[\left(\frac{\hbar_4^\circ}{2\hbar_4}\right)^\circ - \frac{\hbar_4^\circ}{2\hbar_4}\frac{\hbar_4^\circ}{2\hbar_4}\right] \\ &\quad + \frac{\partial_k h_4}{2h_4}\frac{\hbar_4^\circ}{2\hbar_4} + \frac{\partial_k\hbar_4}{2\hbar_4}\frac{\hbar_4^\circ}{2\hbar_4} - \frac{\partial_k\hbar_4^\circ}{2\hbar_4} + \frac{\hbar_4^\circ\partial_k\hbar_4}{2(\hbar_4)^2}, \\ [3]R_{4k} &= w_k\left(\frac{h_4^*}{2h_4}\frac{\hbar_4^\circ}{2\hbar_4}\right) + n_k\left(\frac{\hbar_4^\circ}{2\hbar_4}\right)^2 - \frac{\partial_k h_4}{2h_4}\frac{\hbar_4^\circ}{2\hbar_4} - \frac{\hbar_4^\circ}{2\hbar_4}\frac{\partial_k\hbar_4}{2\hbar_4}. \end{aligned}$$

We summarize above three terms,

$$\begin{aligned}\hat{R}_{4k} = & -n_k^{**} \frac{h_4}{2h_3} \underline{h}_4 + n_k^* \left(-\frac{h_4^*}{2h_3} + \frac{h_4^* h_3^*}{2(h_3)^*} - \frac{h_4^* h_3^*}{4(h_3)^*} - \frac{h_4^*}{4h_3} \right) \underline{h}_4 \\ & + n_k \left(-\frac{\underline{h}_4^{\circ\circ}}{2\underline{h}_4} + \frac{(\underline{h}_4^{\circ})^2}{2(\underline{h}_4)^2} + \left(\frac{\underline{h}_4^{\circ}}{2\underline{h}_4} \right)^{\circ} - \frac{\underline{h}_4^{\circ}}{2\underline{h}_4} \frac{\underline{h}_4^{\circ}}{2\underline{h}_4} + \left(\frac{\underline{h}_4^{\circ}}{2\underline{h}_4} \right)^2 \right) + \frac{\partial_k \underline{h}_4^{\circ}}{2\underline{h}_4} - \frac{\underline{h}_4^{\circ} \partial_k \underline{h}_4}{2(\underline{h}_4)^2} \\ & + \frac{\partial_k h_4}{2h_4} \frac{\underline{h}_4^{\circ}}{2\underline{h}_4} + \frac{\partial_k \underline{h}_4}{2\underline{h}_4} \frac{\underline{h}_4^{\circ}}{2\underline{h}_4} - \frac{\partial_k \underline{h}_4^{\circ}}{2\underline{h}_4} + \frac{\underline{h}_4^{\circ} \partial_k \underline{h}_4}{2(\underline{h}_4)^2} - \frac{\partial_k h_4}{2h_4} \frac{\underline{h}_4^{\circ}}{2\underline{h}_4} - \frac{\underline{h}_4^{\circ}}{2\underline{h}_4} \frac{\partial_k \underline{h}_4}{2\underline{h}_4},\end{aligned}$$

and prove equations (27).

For $\hat{R}_{jka}^i = \frac{\partial \hat{L}_{jk}^i}{\partial y^k} - \left(\frac{\partial \hat{C}_{ja}^i}{\partial x^k} + \hat{L}_{lk}^i \hat{C}_{ja}^l - \hat{L}_{jk}^l \hat{C}_{la}^i - \hat{L}_{ak}^c \hat{C}_{jc}^i \right) + \hat{C}_{jb}^i \hat{T}_{ka}^b$ from (A.6), we obtain zero values because $\hat{C}_{jb}^i = 0$ and \hat{L}_{jk}^i do not depend on y^k . So, $\hat{R}_{ja} = \hat{R}_{jia} = 0$.

Taking $\hat{R}_{bcd}^a = \frac{\partial \hat{C}_{bc}^a}{\partial y^d} - \frac{\partial \hat{C}_{bd}^a}{\partial y^c} + \hat{C}_{bc}^e \hat{C}_{ed}^a - \hat{C}_{bd}^e \hat{C}_{ec}^a$ from (A.6) and contracting the indices in order to obtain the Ricci coefficients, $\hat{R}_{bc} = \frac{\partial \hat{C}_{bc}^d}{\partial y^d} - \frac{\partial \hat{C}_{bd}^d}{\partial y^c} + \hat{C}_{bc}^e \hat{C}_e - \hat{C}_{bd}^e \hat{C}_{ec}^d$, we compute $\hat{R}_{bc} = (\hat{C}_{bc}^3)^* + (\hat{C}_{bc}^4)^{\circ} - \partial_c \hat{C}_b + \hat{C}_{bc}^3 \hat{C}_3 + \hat{C}_{bc}^4 \hat{C}_4 - \hat{C}_{b3}^3 \hat{C}_{3c}^3 - \hat{C}_{b4}^3 \hat{C}_{3c}^4 - \hat{C}_{b3}^4 \hat{C}_{4c}^3 - \hat{C}_{b4}^4 \hat{C}_{4c}^4$. There are nontrivial values,

$$\begin{aligned}\hat{R}_{33} &= (\hat{C}_{33}^3)^* + (\hat{C}_{33}^4)^{\circ} - \hat{C}_3^* + \hat{C}_{33}^3 \hat{C}_3 + \hat{C}_{33}^4 \hat{C}_4 - \hat{C}_{33}^3 \hat{C}_{33}^3 - 2\hat{C}_{34}^3 \hat{C}_{33}^4 - \hat{C}_{34}^4 \hat{C}_{43}^4 \\ &= \left(\frac{h_3^*}{2h_3} \right)^* - \left(\frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4} \right)^* + \frac{h_3^*}{2h_3} \left(\frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4} \right) - \left(\frac{h_3^*}{2h_3} \right)^2 - \left(\frac{h_4^*}{2h_4} \right)^2 \\ &= -\frac{1}{2} \frac{h_4^{**}}{h_4} + \frac{1}{4} \frac{(h_4^*)^2}{(h_4)^2} + \frac{1}{4} \frac{h_3^* h_4^*}{h_3 h_4}, \\ \hat{R}_{44} &= (\hat{C}_{44}^3)^* + (\hat{C}_{44}^4)^{\circ} - \partial_4 \hat{C}_4 + \hat{C}_{44}^3 \hat{C}_3 + \hat{C}_{44}^4 \hat{C}_4 - \hat{C}_{43}^3 \hat{C}_{34}^3 - 2\hat{C}_{44}^3 \hat{C}_{34}^4 - \hat{C}_{44}^4 \hat{C}_{44}^4 \\ &= -\frac{1}{2} \left(\frac{h_4^*}{h_3} \right)^* \frac{\underline{h}_4}{\underline{h}_3} - \frac{h_4^* \underline{h}_4}{2h_3 \underline{h}_3} \left(\frac{h_3^*}{2h_3} + \frac{h_4^*}{2h_4} - \frac{h_4^*}{h_4} \right) \\ &= -\frac{1}{2} \frac{h_4^{**}}{h_3} \underline{h}_4 + \frac{1}{4} \frac{h_3^* h_4^*}{(h_3)^2} \underline{h}_4 + \frac{1}{4} \frac{h_4^*}{h_3} \frac{h_4^*}{h_4} \underline{h}_4.\end{aligned}$$

We get the nontrivial v-coefficients of the Ricci d-tensor,

$$\begin{aligned}\hat{R}_3^3 &= \frac{1}{h_3 \underline{h}_3} \hat{R}_{33} = \frac{1}{2h_3 h_4} [-h_4^{**} + \frac{(h_4^*)^2}{2h_4} + \frac{h_3^* h_4^*}{2h_3}] \frac{1}{\underline{h}_3}, \\ \hat{R}_4^4 &= \frac{1}{h_4 \underline{h}_4} \hat{R}_{44} = \frac{1}{2h_3 h_4} [-h_4^{**} + \frac{(h_4^*)^2}{2h_4} + \frac{h_3^* h_4^*}{2h_3}] \frac{1}{\underline{h}_3},\end{aligned}$$

i.e. the equations (25).

B.4 Zero torsion conditions

We analyze how to solve the equation

$$\widehat{T}_{4k}^4 = \widehat{L}_{4k}^4 - e_4(N_k^4) = \frac{\partial_k(h_4 \underline{h}_4)}{2h_4 \underline{h}_4} - w_k \frac{h_4^*}{2h_4} - n_k \frac{\underline{h}_4^\circ}{2\underline{h}_4} = 0,$$

which follows from formulas (B.5) for a vanishing torsion for $\widehat{\mathbf{D}}$. Taking any \underline{h}_4 for which

$$n_k \underline{h}_4^\circ = \partial_k \underline{h}_4, \quad (\text{B.6})$$

the condition $n_k \frac{h_4^*}{2h_4} \frac{\underline{h}_4^\circ}{2\underline{h}_4} - \frac{h_4^*}{2h_4} \frac{\partial_k \underline{h}_4}{2\underline{h}_4} = 0$ is satisfied. For instance, parametrizing $\underline{h}_4 = {}^h \underline{h}_4(x^k) \underline{h}(y^4)$, the equation (B.6) is solved for any

$$\underline{h}(y^4) = e^{\varkappa y^4} \text{ and } n_k = \varkappa \partial_k [{}^h \underline{h}_4(x^k)], \text{ for } \varkappa = \text{const.}$$

We conclude that for any n_k and \underline{h}_4 related by conditions (B.6) the zero torsion conditions (B.5) are the same as for $\underline{h}_4 = \text{const.}$ Using a similar proof from [10, 11], it is possible to verify by straightforward computations that $\widehat{T}_{\beta\gamma}^\alpha = 0$ if the equations (15) are solved.

B.5 Geometric data for diagonal EYMH configurations

The diagonal ansatz for generating solutions of the system (30)–(32) can be written in the form

$$\begin{aligned} {}^\circ \mathbf{g} &= {}^\circ g_i(x^1) dx^i \otimes dx^i + {}^\circ h_a(x^1, x^2) dy^a \otimes dy^a = \\ &= q^{-1}(r) dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2 \theta d\varphi \otimes d\varphi - \sigma^2(r) q(r) dt \otimes dt, \end{aligned} \quad (\text{B.7})$$

where the coordinates and metric coefficients are parametrized, respectively,

$$\begin{aligned} u^\alpha &= (x^1 = r, x^2 = \theta, y^3 = \varphi, y^4 = t), \\ {}^\circ g_1 &= q^{-1}(r), {}^\circ g_2 = r^2, {}^\circ h_3 = r^2 \sin^2 \theta, {}^\circ h_4 = -\sigma^2(r) q(r) \end{aligned}$$

for $q(r) = 1 - 2m(r)/r - \Lambda r^2/3$, where Λ is a cosmological constant. The function $m(r)$ is usually interpreted as the total mass-energy within the radius r which for $m(r) = 0$ defines an empty de Sitter, dS , space written in a static coordinate system with a cosmological horizon at $r = r_c = \sqrt{3/\Lambda}$. The solution of Yang–Mills equations (30) associated to the quadratic metric element (B.7) is defined by a single magnetic potential $\omega(r)$,

$${}^\circ A = {}^\circ A_2 dx^2 + {}^\circ A_3 dy^3 = \frac{1}{2e} [\omega(r) \tau_1 d\theta + (\cos \theta \tau_3 + \omega(r) \tau_2 \sin \theta) d\varphi], \quad (\text{B.8})$$

where τ_1, τ_2, τ_3 are Pauli matrices. The corresponding solution of (32) is given by

$$\Phi = {}^\circ\Phi = \varpi(r)\tau_3. \quad (\text{B.9})$$

Explicit values for the functions $\sigma(r), q(r), \omega(r), \varpi(r)$ have been found, for instance, in Ref. [18] following certain considerations that the data (B.7), (B.8) and (B.9), i.e. $[{}^\circ\mathbf{g}(r), {}^\circ A(r), {}^\circ\Phi(r)]$, define physical solutions with diagonal metrics depending only on radial coordinate. A well known diagonal Schwarzschild–de Sitter solution of (30)–(32) is that given by data

$$\omega(r) = \pm 1, \sigma(r) = 1, \phi(r) = 0, q(r) = 1 - 2M/r - \Lambda r^2/3$$

which defines a black hole configuration inside a cosmological horizon because $q(r) = 0$ has two positive solutions and $M < 1/3\sqrt{\Lambda}$.

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